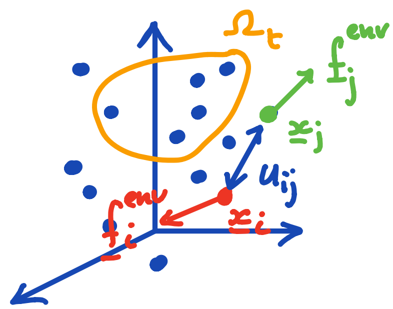
Balance Laws

Important ideas:

1) Balance of mass, momentum, energy and entropy
2) Difference between integral and local form and between Eulerian and Lagrangian forms
3) Axiom of material frame indifference

Motivation

Consider system of N particles (atoms)



m_i : masses

$\underline{x}_i$: positions

$U_{ij} = U_{ji}$: interaction energy

$\underline{f}_{ij}^{int} = -\nabla_x U_{ij}$: interaction force

$\underline{f}_i^{env}$: environmental force

$$\left.\begin{aligned} &\text{Mass is constant: } \dot{m}_i = 0 \\ &\text{Newton's 2}^{nd}\text{ law: } m_i \ddot{\underline{x}}_i = \underline{f}_i^{env} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \underline{f}_{ij}^{int} \end{aligned}\right\} \quad \forall i = 1, \dots, N$$

To any subset Ω_t of particles $I=\{2, 7, \ldots, 17\}$

total mass : $M[\Omega_t]=\sum_{i\in I} m_i$

linear momentum: $\underline{L}[\Omega_t] = \sum_{i\in I} m_i \dot{\underline{x}}_i$

angular momentum: $\underline{j}[\Omega_t] = \sum_{i\in I} \underline{x}_i \times m_i \dot{\underline{x}}_i$

internal energy : $U[\Omega_t] = \sum_{\substack{i,j\in I \\ i>j}} u_{ij}$

kinetic energy: $K[\Omega_t] = \sum_{i\in I} \frac{1}{2} m_i |\dot{\underline{x}}_i|^2$

We have the following balance laws:

Mass is conserved: $\frac{d}{dt} M[\Omega_t] = 0$

Change in linear & angular momentum is equal to the resultant external force & torque on Ω_t

$$\frac{d}{dt} \underline{L}[\Omega_t] = \sum_{i\in I} \left[\underline{f}_i^{env} + \sum_{j\notin I} \underline{f}_{ij}^{int} \right]$$

$$\frac{d}{dt} \underline{j}[\Omega_t] = \sum_{i\in I} \underline{x}_i \times \left[\underline{f}_i^{env} + \sum_{j\notin I} \underline{f}_{ij}^{int} \right]$$

Change in internal and kinetic energy is due to the power of external forces

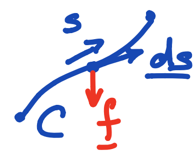
$$\frac{d}{dt}\left(U[\Omega_t] + K[\Omega_t]\right) = \sum_{i\in I} \dot{\underline{x}}_i \cdot \left[\underline{f}_i^{env} + \sum_{j\notin I} \underline{f}_{ij}^{int} \right]$$

Reminder:

Work is energy transferred by application of force along distance

$W = F s$ or

$$W = \int_C \underline{f} \cdot d\underline{s} = \int_{t_1}^{t_2} \underline{f} \cdot \frac{d\underline{s}}{dt}\, dt = \int_{t_1}^{t_2} \underline{f} \cdot \underline{v}\, dt$$



Power is the rate of work

$$\boxed{P = \frac{dW}{dt} = \underline{f} \cdot \underline{v}} \qquad \text{from} \qquad W = \int_{t_1}^{t_2} \frac{dW}{dt}\, dt = \int_{t_1}^{t_2} \underline{f} \cdot \underline{v}\, dt$$

To generalize balance laws to continuum: $\Sigma \rightarrow \int$

$\Rightarrow$ mass, linear & angular momentum

Continuum energy balance is more complicated as we lose information about velocity fluctuations. Continuum velocity gives only the mean velocity.

$\Rightarrow$ Introduce new variables:

Temperature - measure magnitude of velocity fluctuations

Heat - measure the energy in fluctuations

Balance laws in integral form

We define the mass, linear and angular momentum of $\Omega_t \subset B_t$ as:

$$M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_x$$

$$\underline{L}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, \underline{v}(\underline{x},t)\, dV_x$$

$$\underline{j}[\Omega_t]_z = \int_{\Omega_t} (\underline{x}-\underline{z}) \times \rho(\underline{x},t)\, \underline{v}(\underline{x},t)\, dV_x$$

Conservation of mass

In the absence of reactions or relativistic effects, the mass of any subset Ω_t of a continuum body does not change as the body changes shape and place

$$\boxed{\frac{d}{dt} M[\Omega_t] = 0} \qquad \text{for all } \Omega_t \subseteq B_t$$

Laws of inertia

With respect to a fixed frame of reference, the rate of change of linear and angular momentum of any $\Omega_t \subseteq B_t$ equal the resultant force & torque about the origin

$$\boxed{\begin{aligned} \frac{d}{dt} \underline{L}[\Omega] &= \int_{\Omega_t} \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{t}(\underline{x},t)\,dA_x \\ \frac{d}{dt} \underline{j}[\Omega]_0 &= \int_{\Omega_t} \underline{x} \times \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{x} \times \underline{t}(\underline{x},t)\,dA_x \end{aligned}}$$

Continuum Thermodynamics

Temperature and heat:

We assume existence of an absolut temperature field $\theta(\underline{x},t) > 0$ at all $\underline{x} \in B_t$. It is a measure of the velocity of fluctuations of atoms in vicinity of $\underline{x}$.

Thermal energy or heat content is the energy associated with temperature, i.e., velocity fluctuations. Bodies can exchange heat and mechanical work. Heat can be gained or lost in two ways:

I) Body heating: $Q_b[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, r(\underline{x},t)\, dV_x$

II) Surface heating: $Q_s[\Omega_t] = -\int_{\partial\Omega_t} \underbrace{\underline{q}(\underline{x},t) \cdot \underline{n}(\underline{x},t)}_{-h(\underline{x},t)}\, dA_x$

where $r(\underline{x},t)$ is heat supply/loss per unit mass and $\underline{q}(\underline{x},t)$ is the heat flux vector.

Net heating is the rate at which heat is added or lost from Ω_t

$$\boxed{Q[\Omega_t] = Q_b[\Omega_t] + Q_s[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, r(\underline{x},t)\, dV_x - \int_{\partial\Omega_t} \underline{q}(\underline{x},t)\cdot\underline{n}(\underline{x},t)\, dA_x}$$

Kinetic Energy of a continuum body is

$$K[\Omega_t] = \int_{\Omega_t} \frac{1}{2}\rho(\underline{x},t)\, |\underline{v}(\underline{x},t)|^2\, dV_x$$

Power of external forces acting on Ω_t is

$$P[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, \underline{b}(\underline{x},t)\cdot\underline{v}(\underline{x},t)\, dV_x + \int_{\partial\Omega_t} \underline{t}(\underline{x},t)\cdot\underline{v}(\underline{x},t)\, dA_x$$

Net working $W[\Omega_t]$ of external forces on Ω_t is the mechanical power not converted into motion

$$W[\Omega_t] = P[\Omega_t] - \frac{d}{dt} K[\Omega_t]$$

$W[\Omega_t] > 0$: mechanical energy is stored

$W[\Omega_t] < 0$: stored energy is released

Internal energy and The First Law

Energy content of a body not associated with kinetic energy is called internal energy. Here we assume internal energy consists only of thermal (heat) and mechanical (elastic) energy. We neglect electromagnetic and chemical energy.

The internal energy $U[\Omega_t]$ of $\Omega_t \subseteq B$ is given by

$$U[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\,\phi(\underline{x},t)\,dV_x$$

where $\phi(\underline{x},t)$ is the internal energy density field per unit mass.

First Law of Thermodynamics

$$\boxed{\frac{d}{dt} U[\Omega_t] = Q[\Omega_t] + W[\Omega_t]} \qquad \text{for all } \Omega_t \subseteq B_t$$

or

$$\frac{d}{dt}\left(U[\Omega_t] + K[\Omega_t]\right) = Q[\Omega_t] + P[\Omega_t]$$

Fundamental differences between energy balance in particle & continuum systems.

In some cases the power of an external force can be written as $P[\Omega_t] = -\frac{d}{dt} G[\Omega_t]$

where $G[\Omega_t]$ is called potential energy for ext. forces.

$$\Rightarrow \quad \frac{d}{dt}\left(U[\Omega_t] + K[\Omega_t] + G[\Omega_t]\right) = Q[\Omega_t]$$

Entropy and The Second Law

The Second Law expresses the fact that a body has a limit on the rate of heat uptake, but has no limit on the rate of heat releasse.

The second law postulates:

$$Q[\Omega_t] \leq \Xi[\Omega_t]$$

an upper bound $\Xi[\Omega_t]$ on the net heating. In the absence of net working $W[\Omega_t] = 0$ we have

$$\frac{d}{dt} U[\Omega_t] = Q[\Omega_t] \leq \Xi[\Omega_t]$$

$\Rightarrow$ the quantity $\Xi[\Omega_t]$ limits rate of energy storage.

The entropy of a body is defined (up to a constant)

$$\boxed{\frac{d}{dt} S[\Omega_t] = \frac{\Xi[\Omega_t]}{\Theta[\Omega_t]}}$$

where $\Theta[\Omega_t]$ is the uniform temp. of Ω_t.

Entropy is a quantity whose rate of change is equal to the upper heating bound per unit temperature.

At the atomistic level entropy is a measure of disorder, the number of atomic configurations compatible with macroscopic variables. In terms of net heating

$$\boxed{\frac{d}{dt} S[\Omega_t] \geq \frac{Q[\Omega_t]}{\Theta[\Omega_t]}}$$

Clausius-Planck Inequality

In thermo books: $dS = \frac{dQ_{rev}}{T}$ and $dS > \frac{dQ}{T}$

The irreversibility of natural processes is shown by

$$\frac{d}{dt} S[\Omega_t] \geq 0 \qquad \text{when} \quad Q[\Omega_t] = 0$$

For a non-homogeneous body we postulate an entropy density field per unit mass, $s(\underline{x}, t)$, so that

$$S[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, s(\underline{x},t)\, dV_x$$

Hence the generalization of the 2^{nd} law to inhomogeneous systems is given by the Clausius-Duhem ineq.

$$\boxed{\frac{d}{dt} S[\Omega_t] \geq \int_{\Omega_t} \frac{\rho(\underline{x},t)\, r(\underline{x},t)}{\theta(\underline{x},t)}\, dV_x - \int_{\partial\Omega_t} \frac{\underline{q}(\underline{x},t)\cdot \underline{n}(\underline{x},t)}{\theta(\underline{x},t)}\, dA_x}$$

This relation places restrictions on the constitutive relations of a body and leads to statements about energy dissipation and the flow of heat.

Note: The precise form of the 2^{nd} law in continuum mechanics is <u>not</u> settled yet !

But all forms lead to same restrictions.