Material Constraints

Experience shows that some materials resist certain types of deformation.

Example: water resists volume changes but is otherwise highly deformable

We model this by placing a-priori constraints on the material

Def.: A continuum body is subject to a material or internal constraint if every motion φ must satisfy an equation of the form

$$\gamma(\underline{F}(\underline{X},t)) = 0 \qquad \text{for all } X \in B,\ t \geq 0$$

Example: Incompressibility

$$\boxed{\gamma(\underline{F}) = \det(\underline{F}) - 1}$$

To obtain the Eulerian statement note that

$$\dot{\gamma} = \frac{d}{dt}\left(\det(\underline{\underline{F}})\right) = \det(\underline{\underline{F}})\, \mathrm{tr}(\underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}}) = 0 \quad (\text{Lecture } 5)$$

$$= \det(\underline{\underline{F}})\, \mathrm{tr}(\nabla_x \underline{v}) = 0 \quad (\text{Lecture } 15)$$

$$= \det(\underline{\underline{F}})(\nabla_x \cdot \underline{v})_m = 0 \quad t \geq 0$$

Note that $\det(\underline{\underline{F}}) = 1 \Rightarrow \boxed{\nabla_x \cdot \underline{v} = 0}$

Stress fields in constrained materials

Material constraints must be maintained by appropriate stresses. We make the following assumption.

Piola-Kirchhoff stress can be decomposed into

$$\boxed{\underline{\underline{P}}(\underline{X},t) = \underline{\underline{P}}^r(\underline{X},t) + \underline{\underline{P}}^a(\underline{X},t)}$$

$\underline{\underline{P}}^a$ is active stress determined by constitutive eqn

$\underline{\underline{P}}^r$ is reactive stress associated with constraint with zero stress power $\underline{\underline{P}}^r : \dot{\underline{\underline{F}}} = 0$

$\Rightarrow$ $\underline{\underline{P}}^r$ is orthogonal to $\dot{\underline{\underline{F}}}$ in standard inner product.

Because the constraint is constant in time

$$\boxed{\dot{\gamma}(\underline{\underline{F}}(\underline{X},t)) = D\gamma(\underline{\underline{F}}(\underline{X},t)) : \dot{\underline{\underline{F}}}(\underline{X},t) = 0}$$ (Lecture 5)

Where $D\gamma(\underline{\underline{F}})$ denotes the derivative of γ at $\underline{\underline{F}}$.

All φ satisfying γ have the property that

$\dot{\underline{\underline{F}}}$ is orthogonal to $D\gamma(\underline{\underline{F}})$

$\Rightarrow$ $\underline{\underline{P}}^r$ is parallel to $D\gamma(\underline{\underline{F}})$

Thus the most general form for $\underline{\underline{P}}^r$ is

$$\boxed{\underline{\underline{P}}^r(\underline{X},t) = q(\underline{X},t)\, D\gamma(\underline{\underline{F}}(\underline{X},t))}$$

where $q(\underline{X},t)$ is a scalar multiplier. It is the unknown part of the reactive stress that <u>enforces</u> the constraint $\gamma(\underline{\underline{F}}(\underline{X},t)) = 0$.

For incompressible materials $\gamma = \det(\underline{\underline{F}}) - 1$

so that $D\gamma(\underline{\underline{F}}) = \det(\underline{\underline{F}})\underline{\underline{F}}^{-T}$ (Lecture 5)

and the reactive stress is

$$\underline{\underline{P}}^r(\underline{X},t) = q(\underline{X},t)\det(\underline{\underline{F}})\underline{\underline{F}}^{-T}$$

using $\underline{\underline{P}} = \det(\underline{\underline{F}})\, \underline{\underline{\sigma}}_m\, \underline{\underline{F}}^{-T}$

$$\cancel{\det(\underline{\underline{F}})}\, \underline{\underline{\sigma}}_m^r\, \cancel{\underline{\underline{F}}^{-T}} = q(\underline{X},t)\, \cancel{\det(\underline{\underline{F}})}\, \cancel{\underline{\underline{F}}^{-T}}$$

$$\underline{\underline{\sigma}}_m^r = q(\underline{X},t)\, \underline{\underline{I}}$$

$\Rightarrow$ reactive Cauchy stress is spherical: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$

Compare to spherical-deviatoric decomposition

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \mathrm{dev}(\underline{\underline{\sigma}})$$

by comparison we have

$$\boxed{\underline{\underline{\sigma}}^r = -p(\underline{x},t)\,\underline{\underline{I}}} \quad \text{where } p(\underline{x},t) = -q_s(\underline{x},t)$$

$\Rightarrow$ The pressure is the (Lagrange) multiplier that enforces the incompressibility constraint $\nabla_x \cdot \underline{v} = 0$ on the motion.

Isothermal considerations

We will consider isothermal ($\theta(\underline{x},t) = \theta_0$) models of fluids and solids at the end of the course.

$\Rightarrow$ the energy balance not relevant

but the entropy inequality still provides constraints on constitutive models!