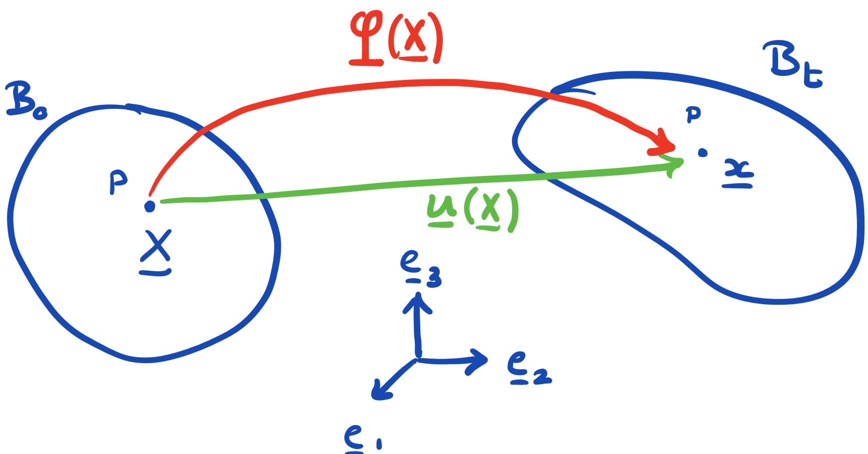
Kinematics

Study of geometry of motion without consideration of mass or stress.

⇒ Quantify the strain and rate of strain.

Deformation Mapping



B_o = body in reference, initial, undeformed or material configuration

B_t = body in current, spatial or deformed config.

p = material point in body

$\underline{X}$ = location of p in B_o

$\underline{\underline{x}}$ = location of p in B_t

$\underline{\varphi}(\underline{X})$ = deformation mapping

$\underline{u}(\underline{X})$ = displacement

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ = frame

$\underline{X} = X_I \, \underline{e}_I$ $\qquad X_I$ = components of $\underline{X}$ in $\{\underline{e}_I\}$

$\underline{\underline{x}} = \underline{x}_i \, \underline{e}_i$ $\qquad x_i$ = " " $\underline{x}$ in $\{\underline{e}_i\}$

Convention:

Upper case quantities & indices → reference. B_0

Lower case quantities & indices → current. B_t

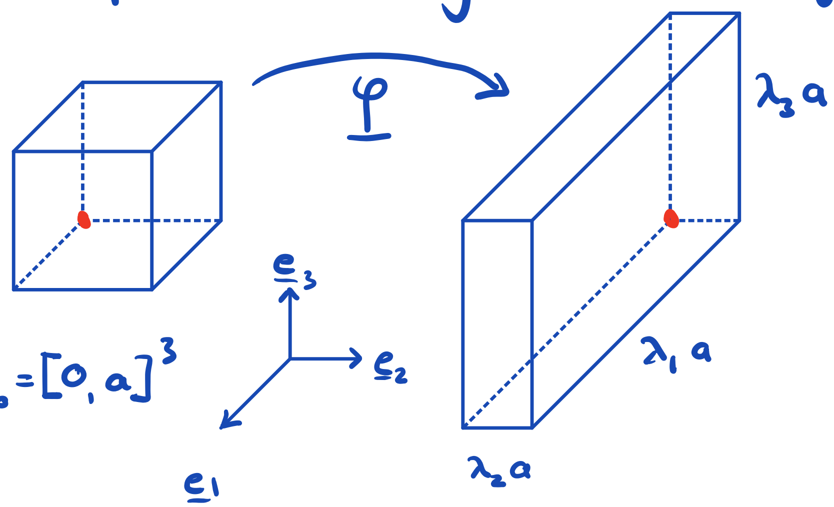
Definition of deformation mapping

$$\boxed{\underline{\underline{x}} = \underline{\varphi}(\underline{X}) = \varphi_i(\underline{X})\, \underline{e}_i}$$

Displacement of a material particle

$$\boxed{\underline{u}(\underline{X}) = \underline{\varphi}(\underline{X}) - \underline{X}}$$

Example: Streching cube with edge length a



$B_0 = [0, a]^3$

deformation map:

$$x_1 = \lambda_1 X_1 + v_1$$

$$x_2 = \lambda_2 X_2 + v_2$$

$$x_3 = \lambda_3 X_3 + v_3$$

λ = strech ratio

$\underline{v}$ = translation (only important in presence of body force)

$(\underline{v} = 0)$

$$\underline{\underline{x}} = \underline{\varphi}(\underline{\underline{X}}) = \lambda_1 X_1 \underline{\underline{e}}_1 + \lambda_2 X_2 \underline{\underline{e}}_2 + \lambda_3 X_3 \underline{\underline{e}}_3 = \Lambda_{ij} X_j \underline{\underline{e}}_i$$

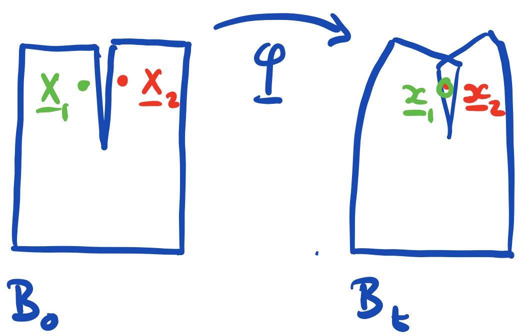
$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\underline{\underline{x}} = \underline{\underline{\Lambda}}\, \underline{\underline{X}}$$

Admissible deformations

For φ to represent the deformation of a body it must satisfy the following conditions:

1) $\underline{\varphi} : B_0 \rightarrow B_t$ is one to one and onto

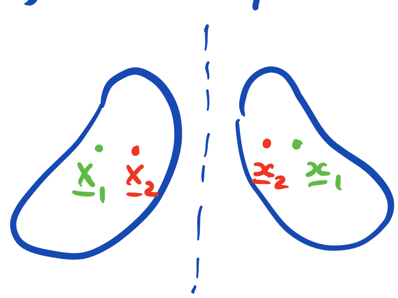


two separate points in B_0 cannot be mapped to same point in B_t.

one to one: for each $\underline{X}$ in B_0 there is at most one $\underline{x}$ in B_t s.t. $\underline{x} = \underline{\varphi}(\underline{X})$

onto: for each $\underline{X}$ in B_0 there is at least one $\underline{x}$ in B_t s.t. $\underline{x} = \underline{\varphi}(\underline{X})$

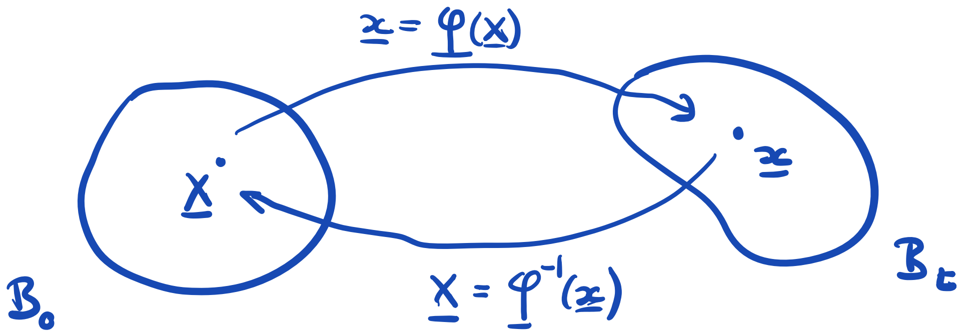
2) $\det(\nabla \underline{\varphi}) > 0$



The orientation of a body is preserved, i.e., a body cannot be deformed into its mirror image.

Inverse Mapping

If $\underline{\varphi}$ is admissible $\Rightarrow$ well defined inverse φ^{-1}

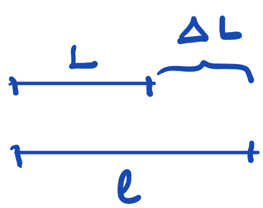


inverse deformation map : $\boxed{\underline{X} = \underline{\varphi}^{-1}(\underline{x})}$

Measures of Strain

In 1D we have simple measures

original:
deformed:



$\Delta L = l - L$

engineering strain: $e = \dfrac{\Delta L}{L} = \dfrac{l - L}{L}$

strech ratio: $\lambda = \dfrac{l}{L}$ $\Rightarrow$ $e = \lambda - 1$

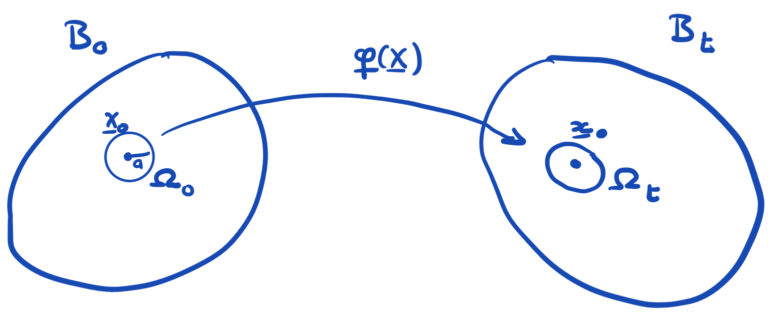
true or Hencky strain: $\varepsilon = \ln(\lambda)$

Green strain: $\varepsilon = \frac{1}{2}(\lambda^2 - 1)$

....

Description of strain is not unique !!

Here we need to find a general 3D approach that is not limited to small deformations.



Sphere Ω_0 of radius a around $\underline{X}_0$.

Mapped to Ω_t around $\underline{x}_0$ by $\varphi(\underline{X})$

$$\Omega_t = \{ \underline{x} \in B_t \mid \underline{x} = \varphi(\underline{X}),\ \underline{X} \in \Omega_0 \} \rightarrow \Omega_t = \varphi(\Omega_0)$$

Def: The strain at $\underline{X}_0$ is any relative difference between Ω_0 and Ω_t in limit of $a \rightarrow 0$.

Deformation gradient

Natural way to quantify local strain

$$\boxed{\underline{\underline{F}}(\underline{X}) = \nabla \underline{\varphi}(\underline{X})} \qquad F_{iJ} = \frac{\partial \varphi_i}{\partial X_J}$$

Expanding deformation in Taylor series around $\underline{X}_0$ we have

$$\underline{\varphi}(\underline{X}) = \underline{\varphi}(\underline{X}_0) + \nabla\underline{\varphi}(\underline{X}_0)(\underline{X} - \underline{X}_0) + O(|\underline{X} - \underline{X}_0|^2)$$

$$= \underbrace{\underline{\varphi}(\underline{X}_0) - \nabla\underline{\varphi}(\underline{X}_0)\underline{X}_0}_{\underline{c}} + \underbrace{\nabla\underline{\varphi}(\underline{X}_0)}_{\underline{\underline{F}}(\underline{X}_0)}\underline{X}$$

locally we can approximat $\underline{\varphi}$ as

$$\boxed{\underline{\varphi}(\underline{X}) \simeq \underline{c} + \underline{\underline{F}}(\underline{X}_0)\underline{X}} \qquad \text{(affine deform.)}$$

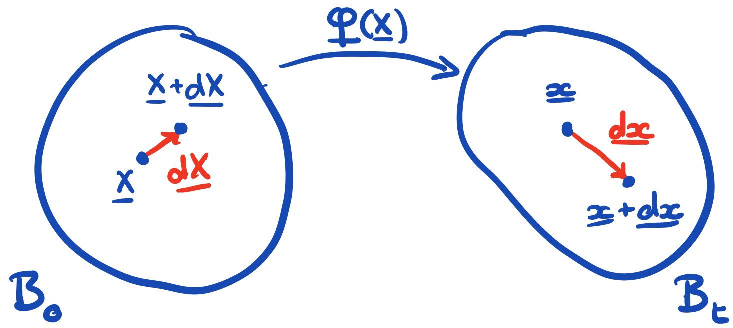
$\Rightarrow$ $\underline{\underline{F}}(\underline{X}_0)$ characterizes local behavior of $\underline{\varphi}(\underline{X})$

Homogeneous deformation

$\underline{\underline{F}}$ is constant

$$\Rightarrow \quad \boxed{\underline{x} = \underline{\varphi}(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}}$$

Consider the mapping of line segment



$|d\underline{X}| \ll 1$

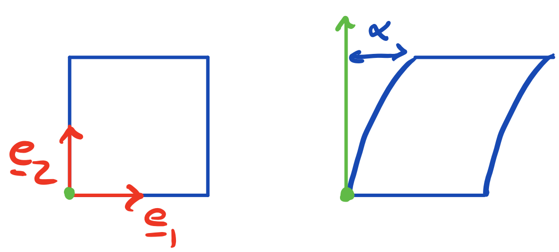
$|d\underline{x}| \ll 1$

$$\underline{x} + d\underline{x} = \underline{\varphi}(\underline{X} + d\underline{X}) \simeq \underline{\varphi}(\underline{X}) + \nabla\underline{\varphi}(\underline{X})\, d\underline{X} = \underline{x} + \underline{\underline{F}}(\underline{X})\, d\underline{X}$$

$$\boxed{\begin{aligned} d\underline{x} &= \underline{\underline{F}}(\underline{X})\, d\underline{X} \\ dx_i &= F_{iJ}(\underline{X})\, dX_J \end{aligned}}$$

$\underline{\underline{F}}$ maps material vectors into spatial vectors.

Example: Shear deformation



$$\underline{\varphi}(\underline{X}) = [X_1 + \alpha X_2^2,\ X_2]$$

$$\nabla\varphi = F = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix}$$

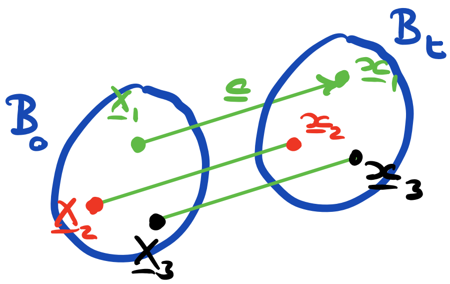
$\underline{\underline{F}}\,\underline{e}_1 = [1, 0]^T = \underline{e}_1$ unchanged

$\underline{\underline{F}}\,\underline{e}_2 = [2\alpha X_2, 1]^T$ rotated and streched

Translations

φ is a translation if $\underline{\underline{F}} = \underline{\underline{I}}$ so that

$$\underline{x} = \underline{c} + \underline{\underline{I}}\,\underline{X} = \underline{c} + \underline{X}$$



$\Rightarrow$ The vector $\underline{c}$ quantifies translation.

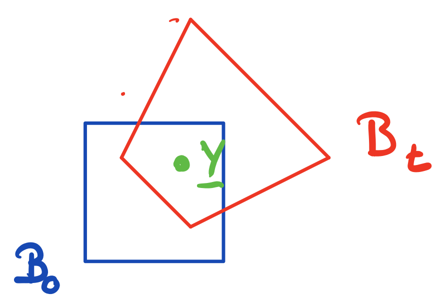
Each point in B_0 is shifted along $\underline{c}$ without change in shape or orientation

Fixed points

Homogeneous deformation has a fixed point at $\underline{Y}$ if

$$\boxed{\varphi(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})}$$

so that $\varphi(\underline{Y}) = \underline{Y}$



Note: The fixed point $\underline{Y}$ must not be in B_0.

Rotations and streches

Homogeneous φ is a rotation about $\underline{y}$ if

$$\boxed{\varphi(\underline{X}) = \underline{y} + \underline{\underline{Q}}(\underline{X} - \underline{y})}$$

for a rotation tensor $\underline{\underline{Q}}$. No change in shape but change in orientation.

Homogeneous φ is a strech about $\underline{y}$ if

$$\boxed{\varphi(\underline{X}) = \underline{y} + \underline{\underline{S}}(\underline{X} - \underline{y})}$$

for any sym. pos. def. tensor $\underline{\underline{S}}$. No change in orientation but change in shape.

Next time: Analysis of local deformation

series of decompositions

I) Translation - Fixed point decomposition

$\underline{\varphi}(\underline{X}) \longrightarrow$ translation & def. with fixed point

II) Polar decomposition

def. with fixed point $\longrightarrow$ rotation & strech

III) Spectral decomposition

strech $\rightarrow$ principal streches

$\Rightarrow$ allows us to formulate strain tensors