Solid Mechanics

Again we neglect thermal effects, so there have 9 governing equations:

$\underline{V} = \dot{\underline{\varphi}}$ 3 kinematic

$\rho_0 \dot{\underline{V}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}$ 3 linear momentum

$\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$ 3 angular momentum

for the following 15 unknowns

$\underline{\varphi} \quad \underline{V} \quad \underline{\underline{P}} \qquad 3 + 3 + 9 = 15$

Again we need $15 - 9 = 6$ additional equations provided by constitutive relations. Here we study functions $\underline{\underline{P}} = \hat{\underline{\underline{P}}}(\underline{\underline{F}})$

$\Rightarrow$ material model is independent of $\underline{V}$

$\Rightarrow$ eliminate $\underline{V}$ by subst. kinematic eqns into lin. mom. balance

Lagrangian stress tensors:

$\underline{\underline{P}} = \det(\underline{\underline{F}})\,\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T} \qquad \underline{\underline{\Sigma}} = \underline{\underline{F}}^{-1}\,\underline{\underline{P}}$

General Elastic Solids

We move from general to specific

1) General elastic materials (Isotropic)
2) Hyper elastic materials
3) Linear elastic materials

A body is said to be an elastic solid if

1) Cauchy stress has form: $\underline{\underline{\sigma}}_m(\underline{X},t) = \hat{\underline{\underline{s}}}_m(\underline{\underline{F}}(\underline{X},t),\underline{X})$

   where $\hat{\underline{\underline{s}}}$ is stress response function

   Stress depends only on present strain

   but not the strain history!

   $\Rightarrow$ generalization of Hooke's Law

2) $\hat{\underline{\underline{s}}}(\underline{\underline{F}},\underline{X})^T = \hat{\underline{\underline{s}}}(\underline{\underline{F}},\underline{X})$ symmetry

   $\Rightarrow$ balance of angular momentum

   is automatically satisfied

A body is homogeneous if $\hat{\underline{\underline{s}}} \neq \hat{\underline{\underline{s}}}(\underline{X})$

Example: St. Venant-Kirchhoff model

$$\boxed{\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \lambda\, \mathrm{tr}(\underline{\underline{E}})\, \underline{\underline{I}} + 2\mu \underline{\underline{E}}}$$ (similar form to Newtonian fl.)

where $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ Green-Lagrange strain tensor

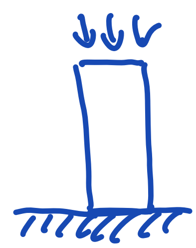
$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ right Cauchy-Green strain tensor

$\lambda, \mu > 0$ are scalar material constants

What are stresses for extreme compression/extension?

Consider uniaxial deformation $\varphi = \begin{pmatrix} X_1 \\ q X_2 \\ X_3 \end{pmatrix}$ $q > 0$.

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$E = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(q^2-1) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

What are the Piola-Kirchhoff stresses?

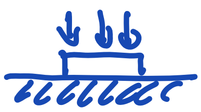
$$\underline{\underline{\Sigma}} = \lambda\, \mathrm{tr}(\underline{\underline{E}})\underline{\underline{I}} + 2\mu \underline{\underline{E}} = \begin{bmatrix} \frac{\lambda}{2}(q^2-1) & 0 & 0 \\ 0 & (\frac{\lambda}{2} + \mu)(q^2-1) & 0 \\ 0 & 0 & \frac{\lambda}{2}(q^2-1) \end{bmatrix}$$

$$\underline{\underline{\Sigma}} = \underline{\underline{F}}^{-1}\underline{\underline{P}} \quad \longrightarrow \quad \underline{\underline{P}} = \underline{\underline{F}}\,\underline{\underline{\Sigma}}$$

$$\underline{\underline{P}} = \begin{bmatrix} \frac{\lambda}{2}(q^2-1) & 0 & 0 \\ 0 & \left(\frac{\lambda}{2}+\mu\right)(q^3-q) & 0 \\ 0 & 0 & \frac{\lambda}{2}(q^2-1) \end{bmatrix}$$

Look at force on the face perp. to deformation

$$\underline{f}_{\underline{e}_2} = \int \underline{\underline{P}}\,\underline{N}_2\, dA_X = \pm\left(\frac{\lambda}{2}+\mu\right)(q^3-q^2)\,\underline{e}_2$$



In the limit of extreme compression $(q \to 0)$ we would expect to have to apply an extreme force!

$$\lim_{q\to 0} |\underline{f}_2| = \lim_{q\to 0} \left(\frac{\lambda}{2}+\mu\right)(q^3-q^2) = 0$$

$\Rightarrow$ clearly St. Venant-Kirchhoff does not apply in this limit.

Elastodynamic Equation

Lagrangian lin. mom. balance (Lecture 19)

$$\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho\, \underline{b}_m$$

replace $\underline{\underline{P}}(\underline{X},t) = \hat{\underline{\underline{P}}}(\underline{\underline{F}}(\underline{X},t))$ where $\underline{\underline{F}} = \nabla \underline{\varphi}$ so that we have a closed system of 3 equations

$$\boxed{\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \hat{\underline{\underline{P}}}(\nabla \underline{\varphi}) + \rho_0 \underline{b}_m}$$

for the 3 unknown components of the motion φ_i.

In index notation with $F_{kl} = \varphi_{k,l}$

$$\nabla_X \cdot \hat{\underline{\underline{P}}} = P_{ij,j} = \frac{\partial \hat{P}_{ij}}{\partial F_{kl}} \frac{\partial F_{kl}}{\partial X_j} \underline{e}_i = \frac{\partial \hat{P}_{ij}}{\partial F_{kl}} \frac{\partial^2 \varphi_k}{\partial X_j \partial X_l} \underline{e}_i$$

so that we have

$$\rho_0 \frac{\partial^2 \varphi_i}{\partial t} = \mathbb{A}_{ijkl} \frac{\partial^2 \varphi_k}{\partial X_j \partial X_k} + \rho_0 b_{m\,i}$$

where $\mathbb{A}_{ijkl} = \partial \hat{P}_{ij} / \partial F_{kl}$

$\Rightarrow$ system second order PDE's for components of $\underline{\varphi}$.

In limit $\ddot{\underline{\varphi}} \to 0$

$$\boxed{\nabla_X \cdot \hat{\underline{\underline{P}}}(\nabla \underline{\varphi}) + \rho_0 \underline{b}_m = 0}$$ Elastostatic Equations

Material Frame Indifference

The Cauchy stress field is only frame-indifferent if the stress response $\hat{\underline{\underline{\sigma}}}$ is written as

$$\boxed{\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{F}}^T}$$

for some function $\bar{\underline{\underline{\sigma}}}$ where $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$. Equivalently the material stress tensors must satisfy

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) \quad \text{and} \quad \hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}})$$

where $\bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) = \sqrt{\det(\underline{\underline{C}})}\;\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})$

To see this consider superposed rigid motion

$$\underline{x}^* = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t)$$

from the axiom of frame indifference (Lecture 20)

$$\underline{\underline{Q}}^T\underline{\underline{\sigma}}^*\underline{\underline{Q}} = \underline{\underline{\sigma}} \quad \text{or} \quad \underline{\underline{Q}}^T\underline{\underline{\sigma}}_m^*\,\underline{\underline{Q}} = \underline{\underline{\sigma}}_m$$

since stress is always given by response function

$$\underline{\underline{\sigma}}_m(\underline{X},t) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}(\underline{X},t)) \quad \text{and} \quad \underline{\underline{\sigma}}_m^* = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}^*(\underline{X},t))$$

note $\hat{\underline{\underline{\sigma}}}$ is independent of ref. frame.

also from axiom of frame indifference $\underline{\underline{F}}^* = \underline{\underline{Q}}\,\underline{\underline{F}}$

$$\Rightarrow \quad \boxed{\underline{\underline{Q}}^T \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{F}})\underline{\underline{Q}} = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})}$$

Polar decomposition $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$ and choose $\underline{\underline{R}} = \underline{\underline{Q}}^T$

so that $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{Q}}^T \underline{\underline{U}})\underline{\underline{R}}^T = \underline{\underline{R}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{U}})\,\underline{\underline{R}}^T$

Define $\underline{\underline{C}}^{1/2} = \sqrt{\underline{\underline{C}}}$, $\underline{\underline{C}}^{-1/2} = (\sqrt{\underline{\underline{C}}})^{-1}$ so that $\underline{\underline{U}} = \underline{\underline{C}}^{1/2}$

and $\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{C}}^{-1/2}$ substituting into $\hat{\underline{\underline{\sigma}}}$ we have

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{F}}^T \quad \text{where} \quad \bar{\underline{\underline{\sigma}}} = \underline{\underline{C}}^{-1/2}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{C}}^{1/2})\,C^{-1/2} \quad \checkmark$$

the results for $\hat{\underline{\underline{P}}}$ and $\hat{\underline{\underline{\Sigma}}}$ follow from definition

and $\det \underline{\underline{C}} = (\det \underline{\underline{F}})^2$.

Implication

Elastic stress response is only frame-indifferent if it depend on φ through $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$. Since $\underline{\underline{F}} = \nabla\varphi$ $\underline{\underline{C}}$ is a non-linear function of φ!

The St. Venant-Kirchhoff model $\hat{\underline{\underline{\Sigma}}} = \lambda \, tr(\underline{\underline{E}}) \underline{\underline{I}} + 2\mu \underline{\underline{E}}$

is frame indifferent after substituting $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$

$$\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) = \frac{\lambda}{2} \, tr(\underline{\underline{C}} - \underline{\underline{I}}) \, \underline{\underline{I}} + \mu (\underline{\underline{C}} - \underline{\underline{I}}) \quad \checkmark$$

Initial Boundary Value Problem

PDE: $\rho_0 \ddot{\varphi} = \nabla_X \cdot \hat{\underline{\underline{P}}}(\nabla\varphi) + \rho_0 \underline{b}_m$ on $\Omega \times [0,T]$

BC: $\varphi = g$ on $\partial\Omega_d \times [0,T]$

$\hat{\underline{\underline{P}}}(\nabla\varphi) \, \underline{N} = \underline{h}$ on $\partial\Omega_\sigma \times [0,T]$

IC: $\varphi(\underline{X}, 0) = X$ on Ω

$\dot{\varphi}(\underline{X}, 0) = V_0$ on Ω

where $\partial\Omega_d \cup \partial\Omega_\sigma = \partial\Omega$ and $\partial\Omega_d \cap \partial\Omega_\sigma = \emptyset$

$\underline{h}$ is force on the boundary

g is a prescribed displacement on boundary

Isotropic Response Function

If the body is isotropic the stress response function takes a simple form. A body is isotropic if

$$\boxed{\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}})} = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}}^T)$$

where $\underline{\underline{Q}}$ a rotation tensor. Isotropy is invariance under rotation to reference configuration.

$\Rightarrow$ has the same stiffness in every direction.

Of course there are anisotropic materials

Need to relate material isotropy to isotropic tensor functions. If stress response of isotropic body is frame indifferent then

$$\tilde{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\tilde{\underline{\underline{\sigma}}}(\underline{\underline{C}})\underline{\underline{Q}}^T \quad \text{and} \quad \tilde{\underline{\underline{\Sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\tilde{\underline{\underline{\Sigma}}}(\underline{\underline{C}})\underline{\underline{Q}}^T$$

ie $\tilde{\underline{\underline{\sigma}}}$ and $\tilde{\underline{\underline{\Sigma}}}$ are isotropic tensor functions.

$\Rightarrow$ deduce frame indifference of $\tilde{\underline{\underline{\sigma}}}$ from frame indifference of $\hat{\underline{\underline{\sigma}}}$

Frame indifferent stress response: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underbrace{\underline{\underline{F}}^T\underline{\underline{F}}}_{\underline{\underline{C}}})\,\underline{\underline{F}}^T$

Isotropic material: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}}^T)$

Combine as follows

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}}^T)$$

$$\underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{F}}^T\underline{\underline{F}})\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{\underline{Q}}^T)\,\underline{\underline{Q}}\,\underline{\underline{F}}^T$$

$$\underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T)\,\underline{\underline{Q}}\,\underline{\underline{F}}^T$$

$$\Rightarrow \quad \bar{\underline{\underline{\sigma}}}(\underline{\underline{C}}) = \underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T)\,\underline{\underline{Q}}$$

$$\underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{Q}} = \bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) \quad \checkmark$$

result for $\bar{\underline{\underline{\Sigma}}}$ follows from definition

Simplified Isotropic Stress Response

For an isotropic body $\hat{\underline{\underline{\sigma}}}$ is frame indifferent only if it can be written in the form

$$\boxed{\begin{aligned}
\hat{\underline{\underline{\sigma}}} &= \underline{\underline{F}}\left[\beta_0(\mathcal{I}_c)\,\underline{\underline{I}} + \beta_1(\mathcal{I}_c)\,\underline{\underline{C}} + \beta_2(\mathcal{I}_c)\,\underline{\underline{C}}^{-1}\right]\underline{\underline{F}}^T \\
\hat{\underline{\underline{P}}} &= \underline{\underline{F}}\left[\gamma_0(\mathcal{I}_c)\,\underline{\underline{I}} + \gamma_1(\mathcal{I}_c)\,\underline{\underline{C}} + \gamma_2(\mathcal{I}_c)\,\underline{\underline{C}}^{-1}\right] \\
\hat{\underline{\underline{\Sigma}}} &= \gamma_0(\mathcal{I}_c)\,\underline{\underline{I}} + \gamma_1(\mathcal{I}_c)\,\underline{\underline{C}} + \gamma_2(\mathcal{I}_c)\,\underline{\underline{C}}^{-1}
\end{aligned}}$$

which follows from $\hat{\underline{\underline{\sigma}}} = \underline{\underline{F}}\, \underline{\underline{\tilde{\sigma}}}(\underline{\underline{C}}) \underline{\underline{F}}^T$

and the fact that $\underline{\underline{\bar{\sigma}}}$ is an isotropic tensor function with the most general form given by second representation theorem (Lecture 21)

$$\underline{\underline{\bar{\sigma}}}(\underline{\underline{C}}) = \beta_0(I_c)\underline{\underline{I}} + \beta_1(I_c)\underline{\underline{C}} + \beta_2(I_c)\underline{\underline{C}}^{-1}$$

The constants $\gamma_i = \sqrt{\det C}\, \beta_i$

Example: St. Venant - Kirchhoff model

$$\underline{\underline{\Sigma}}(\underline{\underline{C}}) = \frac{\lambda}{2} \operatorname{tr}(\underline{\underline{C}} - \underline{\underline{I}})\, \underline{\underline{I}} + \mu(\underline{\underline{C}} - \underline{\underline{I}})$$

$$= \frac{\lambda}{2}\operatorname{tr}(\underline{\underline{C}})\underline{\underline{I}} - \frac{\lambda}{2}\cancel{\operatorname{tr}(\underline{\underline{I}})}^{3}\, \underline{\underline{I}} + \mu\underline{\underline{C}} - \mu\underline{\underline{I}}$$

$$= \left(\frac{\lambda}{2}\operatorname{tr}(\underline{\underline{C}}) - \frac{3\lambda}{2} - \mu\right)\underline{\underline{I}} + \mu\underline{\underline{C}}$$

$$\Rightarrow \quad \gamma_0 = \frac{\lambda}{2}\operatorname{tr}(\underline{\underline{C}}) - \frac{3\lambda}{2} - \mu \qquad \gamma_1 = \mu$$