Fourth-order tensors

By a fourth-order tensor $\mathbb{C}$ we mean a mapping $\mathbb{C}: \mathcal{V}^2 \to \mathcal{V}^2$ which is linear so that:

1) $\mathbb{C}(\underline{\underline{T}} + \underline{\underline{S}}) = \mathbb{C}(\underline{\underline{T}}) + \mathbb{C}(\underline{\underline{S}})$ for all $\underline{\underline{S}}, \underline{\underline{T}} \in \mathcal{V}^2$

2) $\mathbb{C}(\alpha \underline{\underline{S}}) = \alpha\, \mathbb{C}(\underline{\underline{S}})$ for all $\alpha \in \mathbb{R}$ and $\underline{\underline{S}} \in \mathcal{V}^2$

The set of 4-th order tensors is denoted $\mathcal{V}^4$.

Zero tensor: $\mathbb{O}\underline{\underline{T}} = \underline{\underline{0}}$ for all $\underline{\underline{T}} \in \mathcal{V}^2$

Identity tensor: $\mathbb{I}\underline{\underline{T}} = \underline{\underline{T}}$ for all $\underline{\underline{T}} \in \mathcal{V}^2$

Simple example:

For any $\underline{\underline{A}} \in \mathcal{V}^2$ the mapping given by

$$\mathbb{C}(\underline{\underline{T}}) = \underline{\underline{A}}\,\underline{\underline{T}}$$

defines a 4-th tensor. Then for $\alpha, \beta \in \mathbb{R}$ and $\underline{\underline{S}}, \underline{\underline{T}} \in \mathcal{V}^2$

$$\mathbb{C}(\alpha \underline{\underline{S}} + \beta \underline{\underline{T}}) = \underline{\underline{A}}(\alpha \underline{\underline{S}} + \beta \underline{\underline{T}}) = \alpha \underline{\underline{A}}\,\underline{\underline{S}} + \beta \underline{\underline{A}}\,\underline{\underline{T}}$$

$$= \alpha\, \mathbb{C}(\underline{\underline{S}}) + \beta\, \mathbb{C}(\underline{\underline{T}}) \quad \checkmark$$

Forth-order tensor algebra

The set of fourth-order tensors $\mathcal{V}^4$ is a vector space, s.t.

$\mathbb{C} + \mathbb{D} \in \mathcal{V}^4$ and $\alpha \mathbb{C} \in \mathcal{V}^4$ for all $\alpha \in \mathbb{R}$ and $\mathbb{C}, \mathbb{D} \in \mathcal{V}^4$

In addition, we will show that $\mathbb{C}\mathbb{D} \in \mathcal{V}^2$ for all $\mathbb{C}, \mathbb{D} \in \mathcal{V}^4$

We define sum and product of fourth-order tensors

$$(\mathbb{C} + \mathbb{D})\underline{\underline{T}} = \mathbb{C}\underline{\underline{T}} + \mathbb{D}\underline{\underline{T}} \qquad \text{for all } \underline{\underline{T}} \in \mathcal{V}^2$$

$$(\mathbb{C}\mathbb{D})\underline{\underline{T}} = \mathbb{C}(\mathbb{D}\underline{\underline{T}}) \qquad \text{for all } \underline{\underline{T}} \in \mathcal{V}^2$$

Representation of fourth-order tensors

The 81 components of $\mathbb{C}$ in frame $\{\underline{e}_i\}$ are

$$\boxed{\mathbb{C}_{ijkl} = \underline{e}_i \cdot \mathbb{C}(\underline{e}_k \otimes \underline{e}_l)\underline{e}_j}$$

Note that $\mathbb{C}(\underline{e}_k \otimes \underline{e}_l) = \underline{\underline{S}}_{kl}$ is a second order tensor.

so that $\quad \mathbb{C}_{ijkl} = \underline{e}_i \cdot \underline{\underline{S}}_{kl}\, \underline{e}_j \quad$ since $\mathbb{C}$ maps a second order tensor into a second order tensor it can describe the composition of sec. order tensors.

$\mathbb{C}$ as mapping between second-order tensors

Given $\underline{\underline{U}} = U_{ij}\, \underline{e}_i \otimes \underline{e}_j$ and $\underline{\underline{T}} = T_{kl}\, \underline{e}_k \otimes \underline{e}_l$ then what is $\underline{\underline{U}} = \mathbb{C}\,\underline{\underline{T}}$?

$$
\begin{aligned}
U_{ij} &= \underline{e}_i \cdot \underline{\underline{U}}\, \underline{e}_j = \underline{e}_i \cdot \mathbb{C}\, \underline{\underline{T}}\, \underline{e}_j \\
&= \underline{e}_i \cdot \mathbb{C}\left(T_{kl}\, \underline{e}_k \otimes \underline{e}_l\right) \underline{e}_j \\
&= \underbrace{\underline{e}_i \cdot \mathbb{C}\left(\underline{e}_k \otimes \underline{e}_l\right) \underline{e}_j}_{}\, T_{kl}
\end{aligned}
$$

$$\boxed{U_{ij} = \mathbb{C}_{ijkl}\, T_{kl}}$$

$\Rightarrow$ Components of $\mathbb{C}$ are coefficients in linear maping from $\underline{\underline{T}}$ to $\underline{\underline{U}}$

Example: $\mathbb{C}\,\underline{\underline{T}} = \underline{\underline{A}}\,\underline{\underline{T}}$

$$
\begin{aligned}
C_{ijkl} &= \underline{e}_i \cdot \underline{\underline{A}}\left(\underline{e}_k \otimes \underline{e}_l\right) \underline{e}_j \\
&= \underline{e}_i\, \underline{\underline{A}}\, (\underline{e}_l \cdot \underline{e}_j)\, \underline{e}_k = \underline{e}_i\, \underline{\underline{A}}\, \delta_{lj}\, \underline{e}_k \\
&= \underline{e}_i \cdot \underline{\underline{A}}\, \underline{e}_k\, \delta_{lj} = A_{ik}\, \delta_{lj}
\end{aligned}
$$

$$C_{ijkl} = A_{ik}\, \delta_{lj}$$

Product of 2^{nd}-order tensors written as 4^{th}-order tensor.

Fourth-order dyadic products

The dyadic product of four vectors $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ is the fourth order tensor $\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ defined by

$$\boxed{(\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}) \underline{\underline{T}} = (\underline{c} \cdot \underline{\underline{T}} \underline{d}) \underline{a} \otimes \underline{b}}$$

Given frame $\{\underline{e}_i\}$ the set of 81 dyadic products $\{\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l\}$ form a basis for $\mathcal{V}^4$. So that each $\mathbb{C}$ in $\mathcal{V}^4$ can be represented by linear combination

$$\boxed{\mathbb{C} = \mathbb{C}_{ijkl} \, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l}$$

where $\quad \mathbb{C}_{ijkl} = \underline{e}_i \cdot \mathbb{C} (\underline{e}_k \otimes \underline{e}_l) \underline{e}_j$

This gives the correct expression for $\underline{\underline{u}} = \mathbb{C} \underline{\underline{T}}$

$$\begin{aligned}
\underline{\underline{u}} = \mathbb{C} \underline{\underline{T}} &= (\mathbb{C}_{ijkl} \, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l) \underline{\underline{T}} \\
&= \mathbb{C}_{ijkl} (\underline{e}_k \cdot \underline{\underline{T}} \underline{e}_l) \, \underline{e}_i \otimes \underline{e}_j \\
&= \mathbb{C}_{ijkl} T_{kl} \, \underline{e}_i \otimes \underline{e}_j
\end{aligned}$$

$$u_{ij} \, \underline{e}_i \otimes \underline{e}_j = \mathbb{C}_{ijkl} T_{kl} \, \underline{e}_i \otimes \underline{e}_j \quad \Rightarrow \quad u_{ij} = \mathbb{C}_{ijkl} T_{kl} \checkmark$$

Symmetry properties

A fourth order tensor $\mathbb{C} \in \mathcal{V}^4$ is symmetric or has major symmetry if

$$\boxed{\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \mathbb{C}\underline{\underline{A}} : \underline{\underline{B}}} \qquad \text{for all } \underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$$

In components major symmetry implies

$$\boxed{\mathbb{C}_{ijkl} = \mathbb{C}_{klij}}$$

It has a right minor symmetry if

$$\boxed{\begin{gathered} \underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \underline{\underline{A}} : \mathbb{C}\, \mathrm{sym}(\underline{\underline{B}}) \\ \mathbb{C}_{ijkl} = \mathbb{C}_{ijlk} \end{gathered}} \qquad \text{for all } A, B \in \mathcal{V}^2$$

It has a left minor symmetry if

$$\boxed{\begin{gathered} \underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \mathrm{sym}(\underline{\underline{A}}) : \mathbb{C}B \\ \mathbb{C}_{ijkl} = \mathbb{C}_{jikl} \end{gathered}} \qquad \text{for all } \underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$$

To see how component expressions follow from the general definitions, consider:

major sym: $\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = \mathbb{C}\underline{\underline{A}} : \underline{\underline{B}}$

$$\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} = A_{ij}\, C_{ijkl}\, B_{kl} = C_{ijkl} A_{ij}\, B_{kl}$$

$$\mathbb{C}\underline{\underline{A}} : \underline{\underline{B}} = C_{ijkl}\, A_{kl}\, B_{ij} = C_{ijkl}\, A_{kl}\, B_{ij} = C_{klij}\, A_{ij}\, B_{kl}$$

$$\underline{\underline{A}} : \mathbb{C}\underline{\underline{B}} - \mathbb{C}\underline{\underline{A}} : \underline{\underline{B}} = (\mathbb{C}_{ijkl} - \mathbb{C}_{klij})\, A_{ij}\, B_{kl} = 0$$

$$\Rightarrow \quad \mathbb{C}_{ijkl} = \mathbb{C}_{klij} \quad \checkmark$$

similar game for minor symmetries.