Lecture 1: Vectors & Index notation

Review of Vectors

Def: Vector is a quantity with a magnitude & direction

$\underline{v} = |\underline{v}|\,\hat{\underline{v}}$

$|\underline{v}|$ = magnitude $\quad (|\underline{v}| \geq 0)$

$\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$ direction $\quad (|\hat{\underline{v}}| = 1) \qquad$ unit vector

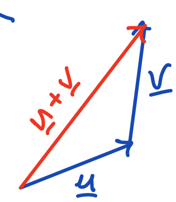
Examples: force, velocities, displacements, ...

Q: Is it possible to have vectors without direction? $\underline{0}$

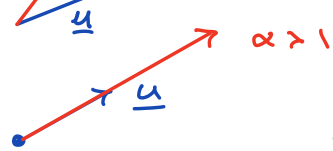
Def: Vector space, $\mathcal{V}$, is a collection of objects that is closed under addition and scalar multiplication.

$\underline{u} \in \mathcal{V} \qquad \underline{v} \in \mathcal{V} \qquad \alpha \in \mathbb{R}$

1) $\underline{u} + \underline{v} \in \mathcal{V}$

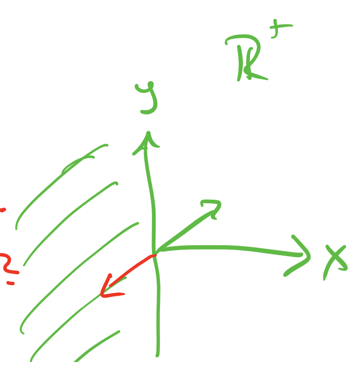


2) $\alpha\, \underline{u} \in \mathcal{V}$

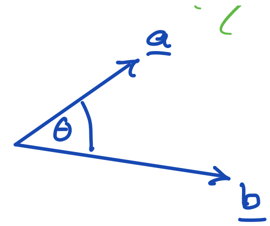


Q1: Do vectors in $\mathbb{R}^3$ form vector space?

Q2: Do vectors in $\mathbb{R}^+$ form vector space?



Scalar product: $\underline{a}, \underline{b} \in \mathcal{V}$



$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta} \qquad \theta \in [0, \pi]$$

$\underline{a} \cdot \underline{b} = 0 \qquad \underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{a} = |\underline{a}|^2$

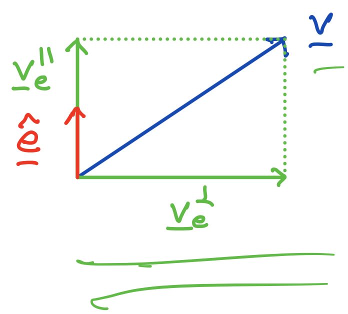
$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a} \qquad$ commutative

Projection: $\hat{\underline{e}}$ unit vector & $\underline{v} \in \mathcal{V}$

$\underline{v} = \underline{v}_e^{\parallel} + \underline{v}_e^{\perp}$

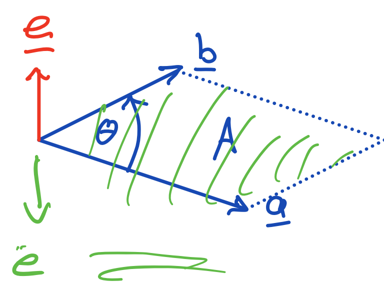
$\underline{v}_e^{\parallel} = \underline{v} \cdot \hat{\underline{e}}$

$\underline{v}_e^{\perp} = \underline{v} - \underline{v}_e^{\parallel}$



Vector product: $\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \; \hat{\underline{e}}} \qquad \theta \in [0, \pi]$$



$\hat{\underline{e}}$ unit vector $\perp$ to $\underline{a}$ & $\underline{b}$

direction right-hand rule

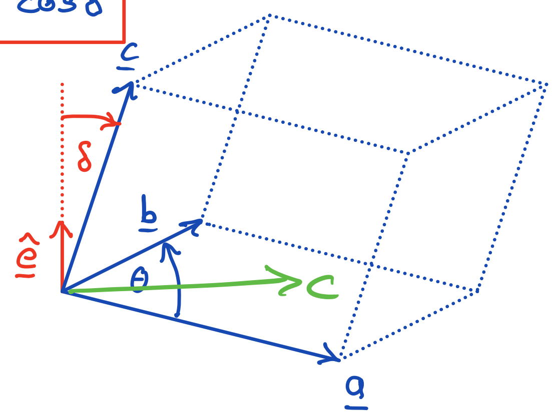
$|\underline{a} \times \underline{b}|$ = Area of paralelogram spanned by $\underline{a}$ & $\underline{b}$

Note: $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a}) \qquad$ not commutative

Q: What does it mean when $\underline{a} \times \underline{b} = \underline{0}$?
$(\underline{a} \neq \underline{0}, \; \underline{b} \neq \underline{0})$

Triple scalar product $\quad \underline{a}, \underline{b}, \underline{c} \in V$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}||\underline{b}||\underline{c}| \sin\theta \cos\delta}$$



θ angle from $\underline{a}$ to $\underline{b}$

$\hat{\underline{e}}$ right handed normal to $\underline{a}$ and b

θ angle from $\hat{\underline{e}}$ to $\underline{c}$

$(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{0} \Rightarrow \underline{a}, \underline{b}, \underline{c}$ linearly dependent

$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$ form right handed system

$(\underline{a} \times \underline{b}) \cdot \underline{c} < 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$ form left handed system

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{b} \times \underline{c}) \cdot \underline{a} = (\underline{c} \times \underline{a}) \cdot \underline{b}}$$

$\Rightarrow$ Volume of parallelepiped spanned by $\underline{a}, \underline{b}, \underline{c}$

Q: $(\underline{a} \times b) \cdot c \overset{?}{=} (\underline{b} \times \underline{a}) \cdot c$

Triple vector product

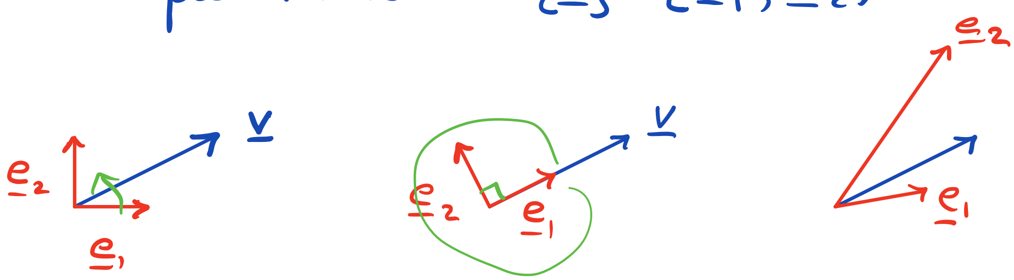
This may be new – we'll talk more about it later

$$\boxed{\begin{aligned} (\underline{a} \times \underline{b}) \times \underline{c} &= (\underline{a} \cdot \underline{c})\underline{b} - (b \cdot \underline{c})\underline{a} \\ \underline{a} \times (\underline{b} \times \underline{c}) &= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \end{aligned}}$$

Basis for a vector space

Def.: Basis for $\mathcal{V}$ is a set of linearly independent vectors $\{\underline{e}\}$ that span the space.

Examples in 2D: $\{\underline{e}\} = \{\hat{\underline{e}}_1, \hat{\underline{e}}_2)$



many choices $\Rightarrow$ not unique

In this class we will always choose a right-handed orthonormal basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

ortho-normal: $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$, $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$, $\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$

right handed: $(\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$

$\Rightarrow$ called Cartesian reference frame

Components of a vector in a basis

Project $\underline{v}$ onto basis vectors to get components.

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 = \sum_{i=1}^{3} v_i \underline{e}_i$$

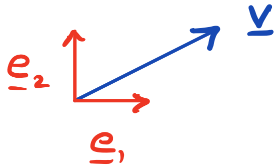
$$v_1 = \underline{v} \cdot \underline{e}_1$$
$$v_2 = \underline{v} \cdot \underline{e}_2$$
$$v_3 = \underline{v} \cdot \underline{e}_3$$

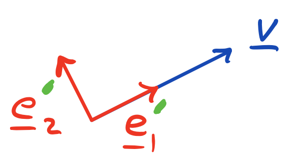
$$[\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Here $[\underline{v}]$ is the representation of $\underline{v}$ in $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

The distinction between a vector and its representation is important for this class.

Example:





$$[\underline{v}] = \begin{bmatrix} \cancel{1} \\ \cancel{2} \end{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix} \qquad [\underline{v}] = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

$$|\underline{v}| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad |\underline{v}| = \sqrt{\sqrt{5}^2 + 0^2} = \sqrt{5}$$

The vector is the same but representation is not.

Index notation

Given ~~to~~ frame $\{\underline{e}_1, \underline{e}_2\ \underline{e}_3\}$

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i \equiv a_i \underline{e}_i$$

Index repeated twice implies summation

"Einstein summation convention"

repeated index "Dummy Index"

Note: symbol is arbitrary $\Rightarrow$ rename indices

$$a_i \underline{e}_i = a_k \underline{e}_k = a_\alpha \underline{e}_\alpha$$

Free index occurs only once!

Example: $a_i = c_j b_j b_i$ $\qquad i$ = free index

$$= \sum_{j=1}^{3} (c_j b_j)\, b_i \qquad j = \text{dummy index}$$

free index is shorthand for set of eqns.

$$a_1 = \sum_{j=1}^{3} (c_j b_j)\, b_1, \quad a_2 = \sum c_j b_j\, b_2, \quad a_3 = \sum c_j b_j\, b_3$$

Basis: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \{\underline{e}_i\}$

Note: • all terms must have same free index

• more than one free index A_{ij}

• cannot use same symbol for dummy & free

• dummy can only be repeated twice

What is wrong with these?

$a_i = b_j$ · not same free index

$$a_i b_j = c_i \underbrace{d_j d_j}_{\text{dummy}}$$

free

$$a_i b_j = c_i c_k e_k d_j + \underbrace{d_p e_e c_e d_q}_{\text{does not have } ij \text{ free ind.}}$$

We'll be back after the break!

Kronecker delta

For any $\{\underline{e}_i\}$ we associate

$$\boxed{\delta_{ij} = \underline{e}_i \cdot \underline{e}_j = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}}$$

$\Rightarrow$ result of orthonormal basis

$\delta_{ij} = \delta_{ji}$ symmetry

$\underline{e}_i = \delta_{ij}\,\underline{e}_j$ transfer property

Example 1: Projection onto basis

$$\underline{u} \cdot \underline{e}_j = (u_i\,\underline{e}_i) \cdot \underline{e}_j = u_i\,\underline{e}_i \cdot \underline{e}_j = u_i\,\delta_{ij} = u_j$$

Example 2: scalar product

$$\underline{a} \cdot \underline{b} = (a_i\,\underline{e}_i) \cdot (b_j\,\underline{e}_j) = a_i b_j\,(\underline{e}_i \cdot \underline{e}_j)$$

$$= a_i b_j\,\delta_{ij} = a_i b_i = a_j b_j = \sum_{i=1}^{3} a_i e_i$$

Kronecker product $\Leftrightarrow$ scalar product

Permutation symbol (Levi-Civita)

Given $\{\underline{e}_i\}$ we associate

$$\boxed{\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = \{123, 231, 312\} \\ -1 & \text{if } ijk = \{321, 213, 132\} \\ 0 & \text{repeated index} \end{cases}}$$

Invariant under cyclic permutation

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312}$$

Flipping any two indices changes the sign

$$\epsilon_{123} = 1 = -\epsilon_{321} = -\epsilon_{213} = -\epsilon_{132}$$

Alternative definitions

$$\epsilon_{ijk} = (\underline{e}_i \times \underline{e}_j) \cdot \underline{e}_k = \det([\underline{e}_i, \underline{e}_j, \underline{e}_k])$$

For ortho normal base

$$\boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k}$$

Vectors product $\quad \underline{a} \times \underline{b} = \underline{c}$

$$\underline{a} = a_i \underline{e}_i \qquad \underline{b} = b_j \underline{e}_j \qquad \underline{c} = c_k \underline{e}_k$$

$$\underline{a} \times \underline{b} = (a_i \underline{e}_i) \times (b_j \underline{e}_j) = c_k \underline{e}_k$$

$$a_i b_j (\underline{e}_i \times \underline{e}_j) = c_k \underline{e}_k$$

$$\epsilon_{ijk} a_i b_j \underline{e}_k = c_k \underline{e}_k$$

$$\boxed{c_k = \epsilon_{ijk} a_i b_j}$$

To express $(\underline{a} \times \underline{b}) \cdot \underline{c}$ in index notation

$$(\epsilon_{ijk} a_i b_j \underline{e}_k) \cdot (c_l \underline{e}_l) = \epsilon_{ijk} a_i b_j c_l (\underline{e}_k \cdot \underline{e}_l)$$

$$= \epsilon_{ijk} a_i b_j c_l \quad \delta_{kL}$$

$$= \epsilon_{ijk} a_i b_j c_k$$

Frame identies

$$\boxed{\underline{e}_i = \delta_{ij} \underline{e}_j} \quad \text{and} \quad \boxed{\underline{e}_i \times \underline{e}_j = \epsilon_{ijk} \underline{e}_k}$$

$\Rightarrow$ consequence of orthonormal basis

Epsilon-delta identities

In any right handed frame we have

$$\boxed{\begin{aligned} \epsilon_{pqs}\,\epsilon_{nrs} &= \delta_{pn}\,\delta_{qr} - \delta_{pr}\,\delta_{qn} \\ \epsilon_{pqs}\,\epsilon_{rqs} &= 2\,\delta_{pr} \end{aligned}}$$

Very useful vector identities

Example: $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\,\underline{b} - (\underline{a}\cdot\underline{b})\,\underline{c} = \underline{d}$

$\underline{a} = a_q\,\underline{e}_q \quad \underline{b} = b_i\,\underline{e}_i \quad \underline{c} = c_j\,\underline{e}_j \qquad \underline{d} = d_p\,\underline{e}_p$

$$\underline{b}\times\underline{c} = \epsilon_{ijk}\, b_i\, c_j\, \underline{e}_k$$

$$(a_q\,\underline{e}_q)\times(\epsilon_{ijk}\, b_i\, c_j\, \underline{e}_k) = \epsilon_{ijk}\, b_i\, c_j\, a_q\, \underbrace{(\underline{e}_q \times \underline{e}_k)}_{\epsilon_{qkp}\,\underline{e}_p}$$

$$\epsilon_{ijk}\,\epsilon_{qkp}\, b_i\, c_j\, a_q\, \underline{e}_p = \epsilon_{ijk}\,\epsilon_{pqk}\, b_i\, c_j\, a_q\, \underline{e}_p$$

$$(\delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp})\, b_i\, c_j\, a_q\, \underline{e}_p$$

First term: $\delta_{ip}\,\delta_{jq}\, b_i\, c_j\, a_q\, \underline{e}_p = b_p\, c_q\, a_q\, \underline{e}_p$

$$= c_q\, a_q\, \underbrace{b_p\, \underline{e}_p}$$

$$= (\underline{c}\cdot\underline{a})\quad \underline{b}$$

Second term: $\delta_{iq}\,\delta_{jp}\, b_i\, c_j\, a_q\, \underline{e}_p =$

$$= b_q\, c_p\, a_q\, \underline{e}_p = \underset{(\underline{c}\cdot\underline{a})}{(b_q a_q)}\; \underbrace{c_p\, \underline{e}_p}_{\underline{c}}$$