$(v \cdot \nabla) v \doteq (\nabla \underline{v}) \underline{v}$

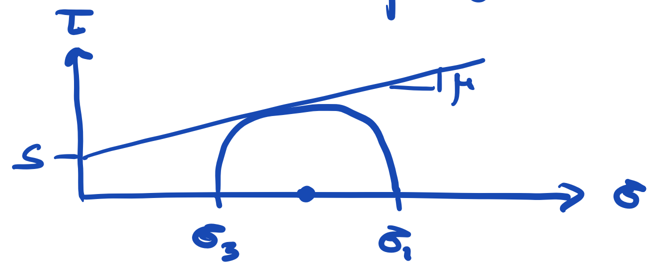
$v \cdot \nabla v$

Lecture 10: Deformation

Logistics: - HW 4 is due Thursday

- Office hrs tomorrow 3pm

Last time: Mohr circle failure



- Simple states of stress:

  hydrostatic, simple & pure shear

  plane stress

→ Done with Stress

Today: Move to Kinematics

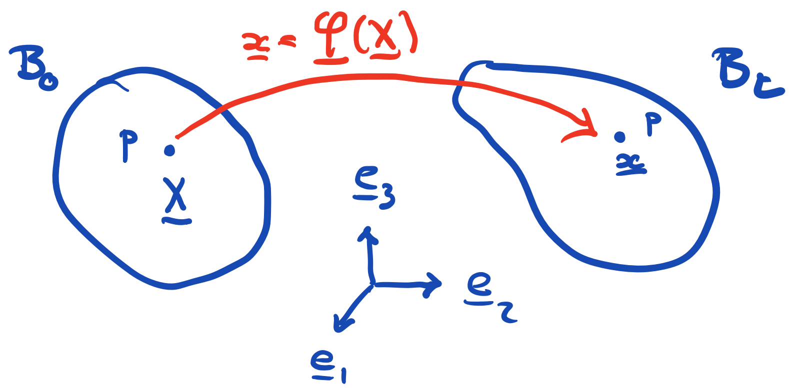
- Deformation mapping
- Measures of Strain
- Deformation gradient

Kinematics

Study of geometry of motion without ~~se~~ considering mass and stress

$\Rightarrow$ quantify strain and rate of strain

Deformation Mapping



$\underline{\underline{X}}$

B_0 = body in reference config.
initial, undeformed, material

B_t = body in current, spatial, deformed config

p = material point in body

$\underline{X}$ = location of p in B_0

$\underline{x}$ = location of p in B_t

$\underline{\varphi}(\underline{X})$ = deformation mapping

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ = frame

$\underline{X} = X_I \underline{e}_I$ $\quad\quad$ X_I = components of $\underline{X}$ in $\{\underline{e}_I\}$

$\underline{x} = x_i \underline{e}_i$ $\quad\quad$ x_i = components of $\underline{x}$ in $\{\underline{e}_i\}$

Convention:

Upper case quantities & indices → ref. conf B_0

Lower case quantitie & " → current conf B_t

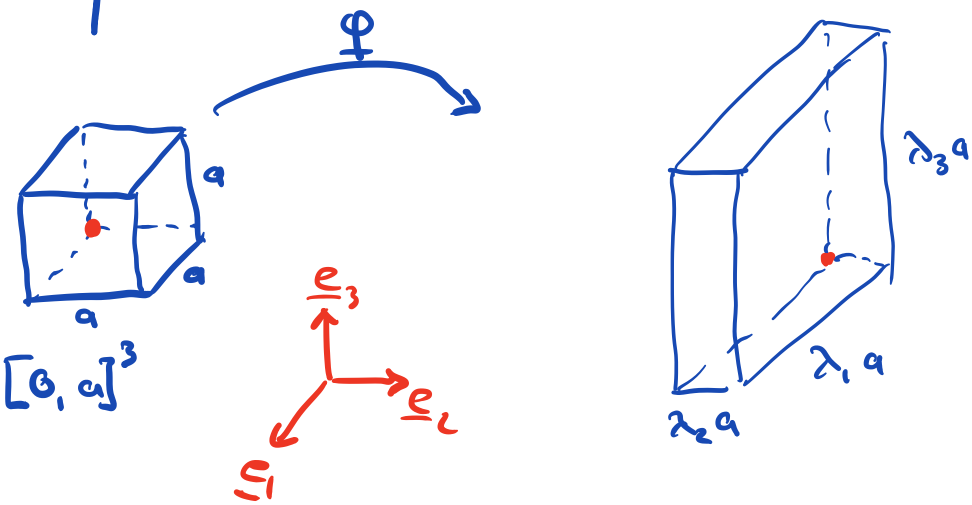
Definition of deformation mapping

$$\underline{x} = \underline{\varphi}(\underline{X}) = \varphi_i(\underline{X})\,\underline{e}_i$$

Displacement of material particle

$$\boxed{\underline{u}(\underline{X}) = \underline{x} - \underline{X} = \underline{\varphi}(\underline{X}) - \underline{X}}$$

Example: Streck unit cube



$B_0 = [0, a]^3$

deformation map:

$$x_1 = \lambda_1 X_1 + v_1$$
$$x_2 = \lambda_2 X_2 + v_2$$
$$x_3 = \lambda_3 X_3 + v_3$$

λ_i: streck ratios

v_i: translations $= 0$

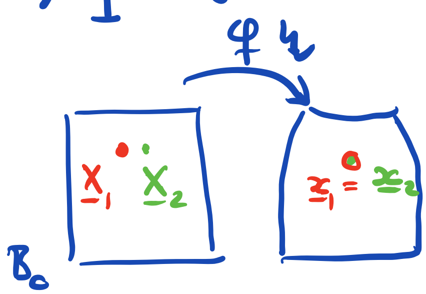
$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

$$\underline{x} = \varphi(\underline{X}) = \lambda_1 X_1 \underline{e}_1 + \lambda_2 X_2 \underline{e}_2 + \lambda_3 X_3 \underline{e}_3$$
$$= \Lambda_{ij} X_j \underline{e}_i = \underline{\underline{\Lambda}}\, \underline{X}$$

Admissible deformations

For φ to represent a physically feasible deformation it must satisfy two following conditions:

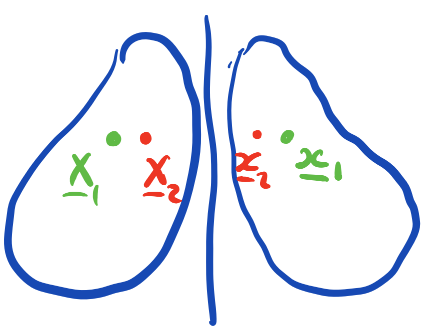
1) $\varphi : B_0 \to B_t$ has to be "one to one and onto"



Two separate material points in B_0 cannot be mapped to same point in B_t.

2) $\det(\nabla\varphi) > 0$ $\quad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

The orientation of a body has to be preserved i.e. no reflections.



Q = is ~~orth~~ orthogonal matrix

$\det(\underline{\underline{R}}\,\underline{\underline{U}}) =$

$\det(\underline{\underline{R}})\,\det(\underline{\underline{U}})$

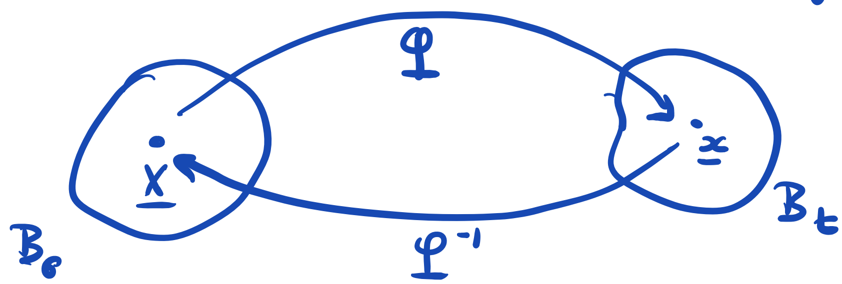
$\underbrace{\lambda_1 \lambda_3 \lambda_3}_{>0}$

$\det(Q) > 1$ rotation

$\det(Q) < 1$ reflection

Inverse Mapping

If φ is admissible $\Rightarrow$ well defined inverse φ^{-1}



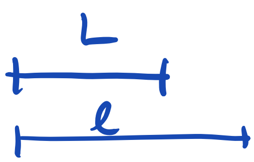
inverse def. map: $\underline{X} = \underline{\varphi}^{-1}(\underline{x})$

Measures of Strain

In 1D we have simple measures

original:

deformed:



$\Delta L = l - L$

engineering strain: $e = \dfrac{\Delta L}{L}$ "relative def."

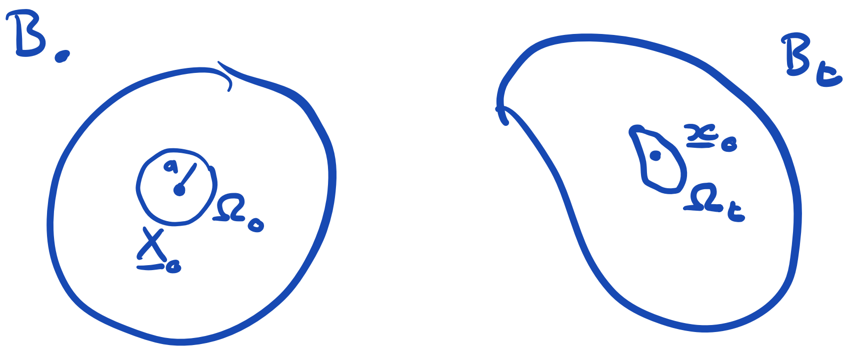
strech ratio : $\lambda = \dfrac{l}{L}$ $\quad e = \lambda - 1$

Hencky strain: $\varepsilon = \ln(\lambda)$

Green strain: $\varepsilon = \frac{1}{2}(\lambda^2 - 1)$

$\Rightarrow$ Quantification of strain is not unique

We are looking for general 3D approach that is not limited to small deformations.



Sphere of radius a around $\underline{X}_0$, is mapped to $\Omega_t = \varphi(\Omega_0)$

$$\Omega_t = \{ \underline{x} \in B_t \mid \underline{x} = \varphi(\underline{X}), \underline{X}_0 \in \Omega_0 \}$$

Def: The strain at $\underline{X}_0$ is any relative difference between Ω_0 and Ω_t in limit of $a \to 0$.

Deformation Gradient

Natural way to quantify strain near $\underline{X}_0$

$$\boxed{\underline{\underline{F}}(\underline{X}) = \nabla \underline{\varphi}(\underline{X})} \qquad F_{iJ} = \frac{\partial \varphi_i}{\partial X_J}$$

Approximate φ using Taylor series around $\underline{X}_0$

$$\underline{\varphi}(\underline{X}) = \underline{\varphi}(\underline{X}_0) + \nabla\underline{\varphi}(\underline{X}_0)(\underline{X} - \underline{X}_0) + \mathcal{O}(|\underline{X} - \underline{X}_0|^2)$$

$$= \underbrace{\underline{\varphi}(\underline{X}_0) - \nabla\varphi(\underline{X}_0)\underline{X}_0}_{c} + \nabla\underline{\varphi}(\underline{X}_0)\,\underline{X}$$

$$= \qquad c \qquad + \underline{\underline{F}}(\underline{X}_0)\,\underline{X}$$

In vicinity of $\underline{X}$ we have lin. approx

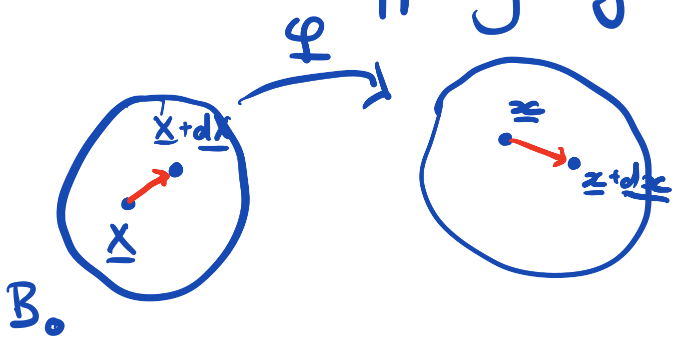
$$\boxed{\underline{\varphi}(\underline{X}) = \underline{c} + \underline{\underline{F}}(\underline{X}_0)\,\underline{X}}$$

$\Rightarrow$ $\underline{\underline{F}}(\underline{X})$ describes local deformation

Homogeneous def. $\quad \underline{\underline{F}}$ is const.

$$\underline{x} = \underline{\varphi}(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X} \qquad \text{affine def.}$$

Consider mapping of a line segment



$|d\underline{X}| \ll 1$

$|d\underline{x}| \ll 1$

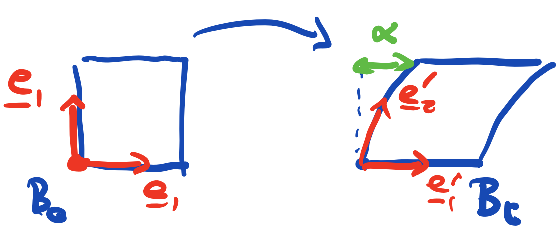
$$\cancel{\underline{x}} + d\underline{x} = \underline{\varphi}(\underline{X}+d\underline{X}) \simeq \underbrace{\underline{\varphi}(\underline{X})}_{\cancel{\underline{x}}} + \underbrace{\nabla\underline{\varphi}(\underline{X})}_{\underline{F}}\, d\underline{X}$$

$$\boxed{\begin{aligned} d\underline{x} &= \underline{F}\, d\underline{X} \\ dx_i &= F_{iJ}\, dX_J \end{aligned}}$$

(hom. def.)

$\underline{F}$ maps material vectors into spatial vectors.

Example: Shear deformation



$$\underline{\varphi}(\underline{X}) = [\underbrace{X_1 + \alpha X_2^2}_{x_1},\ X_2]^T$$

$\frac{\partial \varphi_1}{\partial X_2}$

$$\nabla \varphi = \begin{bmatrix} 1 & 2\alpha X_2 \\ 0 & 1 \end{bmatrix} = \underline{\underline{F}}(\underline{X})$$

$$\underline{\underline{F}}(\underline{0})\underline{e}_1 = [1, 0]^T$$

$\underline{e}_1$ is unchangen

$$\underline{\underline{F}}\,\underline{e}_2 = [2\alpha X_2, 1]^T$$

In admissible deformation