Lecture 11: Local analysis of deformation

Logistics: - HW 4 due

- HW 3 still grading

Do we need HW (2) with more practice on index manipulation?

Last time: - Deformation mapping $\underline{x} = \varphi(\underline{X})$

- Deformation gradient $\underline{\underline{F}} = \nabla \varphi$

$\Rightarrow$ local measure of deformation

Why not use $\underline{\underline{F}}$ as strain tensor?

Today: - Analysis of local deformation

Generally a deformation comprises

1) translation, 2) rotation 3) Strech

only the streck changes shape of body

$\Rightarrow$ Extract strech and build

stress tensor on that

Analysis of local deformation

Any $\varphi(\underline{X})$ can locally be approximated as a hom. affine deform. (Taylor series)

$$\boxed{\underline{x} = \underline{\varphi}(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}} \qquad \text{where } \underline{\underline{F}} = \nabla \varphi$$

$\underline{\underline{F}}$ is a measure of local deformation but not a suitable measure of strain because it contains rotations (and translation)

To build strain tensor we will

1) Remove translations
2) Remove rotations
3) Find principal stretches

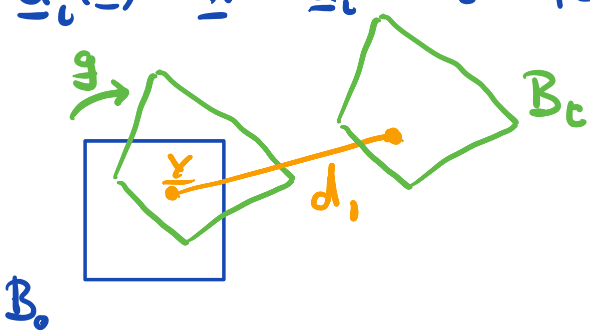
1) Translation – fixed point decomposition

Any hom. φ can be decomposed into

$$\varphi = \underline{d_1 \circ g} = g \circ d_2 = d_1(g(\underline{X})) = g(d_2(\underline{X}))$$

where $g(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ a hom. def. with fixed point $\underline{Y}$ and translations

$\underline{d}_i(\underline{X}) = \underline{X} + \underline{a}_i \qquad i = 1, 2.$



Consider
$$\underline{x} = \underline{c} + \underline{\underline{F}}\,\underline{X}$$
$$\underline{y} = \underline{c} + \underline{\underline{F}}\,\underline{Y}$$

subtract:
$$\underset{\substack{\uparrow \\ \varphi(X)}}{\underline{x}} - \underset{\substack{\uparrow \\ \varphi(Y)}}{\underline{y}} = \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$\boxed{\varphi(\underline{X}) = \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})}$$

similar to Taylor series but for hom. def.

there is no requirement that $|\underline{X} - \underline{Y}| \ll 1$

$$g(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$\underline{d}_i(\underline{X}) = \underline{X} + \underline{a}_i$$

$$\underline{d}_1 \circ g = \underline{d}_1(g(\underline{X})) = g(\underline{X}) + \underline{a}_1$$

$$= \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{a}_1$$

choose $\underline{a}_1 = \varphi(\underline{Y}) - \underline{Y}$ shift of fixed point

$$= \cancel{\underline{Y}} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \varphi(\underline{Y}) - \cancel{\underline{Y}}$$

$$\varphi(\underline{X}) = \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$\varphi(\underline{X}) = (\underline{d}_1 \circ g)(\underline{X})$$

$\Rightarrow$ always extract translation and assume hom. def. with a fixedpoint.

Strech-rotation decomposition

Let $\varphi(\underline{X})$ be a hom. def. with fixed point $\underline{Y}$ so that $\varphi(X) = \underline{Y} - \underline{\underline{F}}(\underline{X} - \underline{Y})$ then we have

$$\varphi = \underset{2}{\underline{r}} \circ \underset{1}{\underline{s}_1} = \underset{2}{\underline{s}_2} \circ \underset{1}{\underline{r}}$$

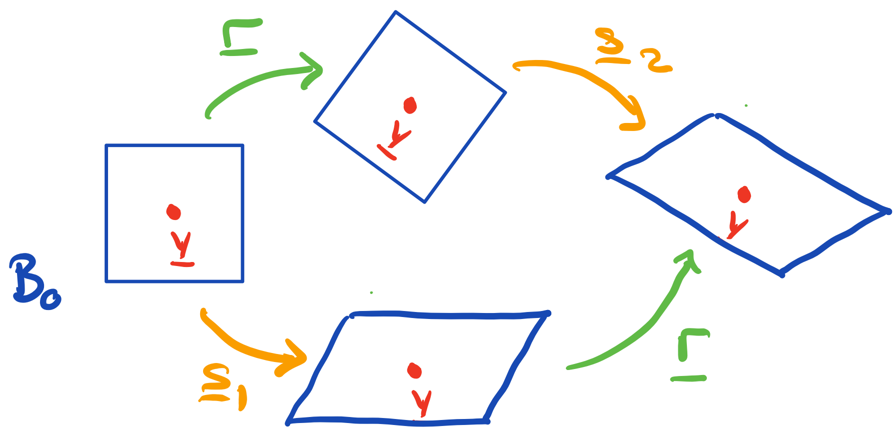
where

$$\underline{r}(\underline{X}) = \underline{Y} + \underline{\underline{R}}(\underline{X}-\underline{Y}) \qquad \text{rotation around } \underline{Y}$$

$$\left.\begin{aligned} \underline{s}_1(\underline{X}) &= \underline{Y} + \underline{\underline{U}}(\underline{X}-\underline{Y}) \\ \underline{s}_2(\underline{X}) &= \underline{Y} + \underline{\underline{V}}(\underline{X}-\underline{Y}) \end{aligned}\right\} \text{ streches from } \underline{Y}$$

The tensors $\underline{\underline{R}}$ $\underline{\underline{U}} = \sqrt{F^T F}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are given by polar decomposition of $\underline{\underline{F}}$

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}} \qquad \rightarrow \text{Lecture 2}$$



To see this consider

$$(\underline{r}\circ\underline{s}_1)(\underline{X}) = \underline{r}(\underline{s}_1(\underline{X})) = \underline{Y} + \underline{\underline{R}}(\underline{s}_1(\underline{X}) - \underline{Y})$$

$$= \underline{Y} + \underline{\underline{R}}\left(\cancel{\underline{Y}} + \underline{\underline{U}}(\underline{X}-\underline{Y}) - \cancel{\underline{Y}}\right)$$

$$= \underline{Y} + \underbrace{\underline{\underline{R}}\,\underline{\underline{U}}}_{\underline{\underline{F}}}(\underline{X}-\underline{Y})$$

$$\underline{r}\circ\underline{s}_1 = \varphi \qquad\qquad =$$

Strech tensors

Both $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d.

$\Rightarrow$ spectral decomposition

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i \qquad \underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i$$

where $\{\lambda_i, \underline{u}_i\}$ and $\{\lambda_i, \underline{v}_i\}$ are eigenpairs of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ respectively.

Note: eigenvalues are same but eigenvec. are not.

$$(\underline{\underline{U}} - \lambda_i I)\,\underline{u}_i = 0$$

$$\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{u}_i$$

any vector $\parallel$ to $\underline{u}_i$ is streched by λ_i

$\lambda_i \rightarrow$ principal streches

$\underline{u}_i$ and $\underline{v}_i$ are right & left princ. directions

$\underline{\underline{U}}$ and $\underline{\underline{V}}$ are right & left princ. strech tensors

Char. polynomial

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$

$$p_u(\lambda) = \det(\underline{\underline{U}} - \lambda \underline{\underline{I}}) \quad \Big\{ \quad \underbrace{\cancel{\underline{\underline{R}}^T\underline{\underline{R}}}}_{}^{\underline{\underline{I}}}\,\underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}}$$

$$= \det(\underline{\underline{R}}^T\underline{\underline{V}}\,\underline{\underline{R}} - \lambda \underline{\underline{R}}^T\underline{\underline{R}}) = \det(\underline{\underline{R}}^T(\underline{\underline{V}} - \lambda \underline{\underline{I}})\,\underline{\underline{R}})$$

$$= \underbrace{\det(\underline{\underline{R}}^T)}_{1}\ \underbrace{\det(\underline{\underline{V}} - \lambda\underline{\underline{I}})}_{p_V(\lambda)}\ \underbrace{\det(\underline{\underline{R}})}_{1}$$

What is relation between $\underline{u}_i$ and $\underline{v}_i$?

$$\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{u}_i$$

$$\underline{\underline{R}}\,\underline{\underline{U}}\,\underline{u}_i = \lambda_i \underline{\underline{R}}\,\underline{u}_i \qquad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$

$$\left.\begin{array}{l} \underline{\underline{V}}\,\underline{\underline{R}}\,\underline{u}_i = \lambda_i \underline{\underline{R}}\,\underline{u}_i \\ \underline{\underline{V}}\,\underline{v}_i = \lambda_i \underline{v}_i \end{array}\right\} \quad \underline{v}_i = \underline{\underline{R}}\,\underline{u}_i$$

In summary:

hom. def $\underline{\varphi}$ can be decomposed into sequence of elementary deformations:

1) Translation $(\underline{d}_i)$

2) Rotation $\underline{\underline{r}}$ 3) Strech $\underline{s}_1$ $\underline{s}_2$

Example: $\underline{\varphi} = \underline{s}_2 \circ \underline{r} \circ \underline{d}_2$

Cauchy-Green Strain Tensor

Consider deformation $\underline{\varphi}: B_0 \to B_c$ with $\underline{\underline{F}} = \nabla \underline{\varphi}$ then the (right) Cauchy-Green strain tensor is

$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2}$$

$\underline{\underline{C}}$ is always s.p.d. by construction.

While $\underline{\underline{F}}$ contains info about both rotation and strech, $\underline{\underline{C}}$ only contains information about streches.

Hence we cannot obtain $\underline{\underline{F}}$ from $\underline{\underline{C}}$!

Remarks: why not just use $\underline{\underline{U}}$

Introduce $\underline{\underline{C}}$ to avoid tensor square root!

Simple example:

$$[\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$[\underline{\underline{C}}] = [\underline{\underline{F}}^T][\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

To get $[U]$ we need to solve eig. prob.

$$\begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 5-\mu & 4 \\ 0 & 4 & 5-\mu \end{vmatrix} = (1-\mu)(5-\mu)^2 - 16(1-\mu) = 0$$

eigen values: $\mu_{1,2} = 1 \quad \mu_3 = 9$

eigen vectors: $[\underline{u}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [\underline{u}_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \underline{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Hence : $[\underline{\underline{U}}] = \sqrt{[\underline{\underline{C}}]} = \sum_{i=1}^{3} \mu_i \, \underline{u}_i \otimes \underline{u}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i$$

$$\underline{\underline{C}} = \underline{\underline{U}}^2 = \sum_{i=1}^{3} \lambda_i^2 \, \underline{u}_i \otimes \underline{u}_2$$

$$\mu_i = \lambda_i^2$$

$\Rightarrow$ eigenvalues of $\underline{\underline{C}}$ are squares of principal streches.

Another option is (left) Cauchy-Green strain t.

$$\underline{\underline{B}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{V}}^2$$

Some solid mechanics considerations

$$\underline{x} = \underline{\underline{F}}\,\underline{X} \qquad x_i = F_{ij} X_j$$

$F \; F$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underbrace{F_{ik} F_{ij}}_{C_{kj}} \, \underline{e}_k \otimes \underline{e}_j$$

"material strain tensor"

$$\underline{\underline{B}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underbrace{F_{iJ}\,F_{kJ}}_{B_{ik}}\ \underline{e}_i \otimes \underline{e}_k$$

"spatial stretches"