Lecture 12: Strain tensors

Logistics: - PS 5 is posted

Q1 is basic index manipulation

→ really for PS 2

- One PS4 is still missing

Last time: - Analysis of local deformation

- Break φ down into

  - translation
  - rotation
  - strech

  $\left.\begin{array}{l}\text{translation}\\ \text{rotation}\end{array}\right\}$ no change in shape

⇒ base strain tensor on strech

- Cauchy-Green: $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{U}}^2$

Today: - Other strain tensors

- Analysis of $\underline{\underline{C}}$

- Volume changes → (class on $\underline{\underline{F}} = \nabla \underline{\varphi}$)

Other strain tensors (Finite strain)

Many ways to measure strain

I) $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ right Cauchy-Green

$C_{KL} = F_{iK} F_{iL}$ material tensor

II) $\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T$ left Cauchy-Green (Finger tensor)

$b_{kl} = F_{kI} F_{lI}$ spatial tensor

III) $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ Green-Lagrange $\Rightarrow$ linear theory

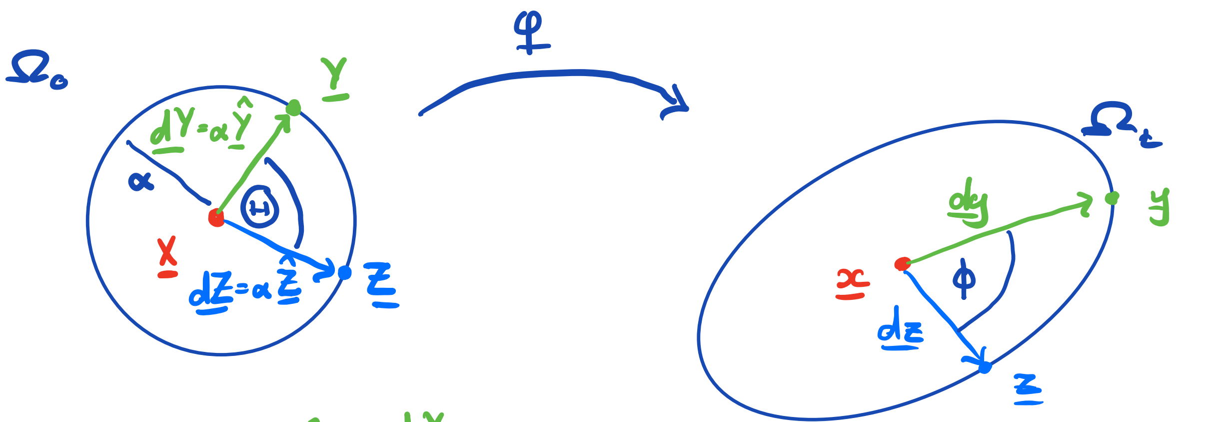
$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})$ material tensor

IV) $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{-T} \underline{\underline{F}}^{-1})$ Euler-Almansi

$e_{kl} = \frac{1}{2}(\delta_{kl} - F^{-1}_{Ik} F^{-1}_{Il})$ spatial tensor

Interpretation of $\underline{\underline{C}}$

How is deformation quantified by comp. $\underline{\underline{C}}$?



$d\underline{Y} = \underline{Y} - \underline{X} \qquad \hat{\underline{Y}} = \dfrac{d\underline{Y}}{|d\underline{Y}|}$

Cauchy-Green strain relations

For any $\underline{X} \in B_0$ and unit vectors $\hat{\underline{Y}}$ and $\hat{\underline{Z}}$ we define $\lambda(\hat{\underline{Y}}) > 0$ and $\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) \in [0, \pi]$ by

$$\boxed{\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Y}}}} \quad \text{and} \quad \boxed{\cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\hat{\underline{Y}}}\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\hat{\underline{Z}}}}}$$

I) Streches

In the limit $\alpha \to 0$

$$\frac{|\underline{y}-\underline{x}|}{|\underline{Y}-\underline{X}|} = \frac{d\underline{y}}{d\underline{Y}} \to \lambda(\hat{\underline{Y}}) \qquad \frac{d\underline{z}}{d\underline{Z}} \to \lambda(\hat{\underline{Z}})$$

$\Rightarrow \lambda(\hat{\underline{Y}})$ is a strech, i.e. ratio of deformed to initial len.

Use $d\underline{y} = \underline{\underline{F}}\, d\underline{Y}$ $\qquad \underline{\underline{A}}\,\underline{v}\cdot\underline{u} = \underline{v}\cdot\underline{\underline{A}}^T\underline{u}$

$$|d\underline{y}|^2 = d\underline{y}\cdot d\underline{y} = \underline{\underline{F}}\,d\underline{Y}\cdot \underline{\underline{F}}\,d\underline{Y} = d\underline{Y}\cdot F^T(\underline{\underline{F}}\,d\underline{Y})$$

$$= d\underline{Y}\cdot F^T F\, d\underline{Y} \qquad d\underline{Y} = \alpha\,\hat{\underline{Y}}$$

$$= \alpha^2\, \hat{\underline{Y}}\cdot\underline{\underline{C}}\,\hat{\underline{Y}}$$

$|d\underline{Y}| = \alpha^2$ by def.

So that $\dfrac{dy^2}{d\underline{Y}^2} = \dfrac{\alpha^2\,\hat{\underline{Y}}\cdot\underline{\underline{C}}\,\hat{\underline{Y}}}{\alpha^2} = \hat{\underline{Y}}\cdot\underline{\underline{C}}\,\hat{\underline{Y}} = \lambda^2(\hat{\underline{Y}})$

If $\underline{U}_i$ is a right-principal strech

$$(\underline{\underline{C}} - \lambda_i^2\,\underline{\underline{I}})\,\hat{\underline{U}}_i = 0 \qquad \text{(no sum)}$$

$$\hat{\underline{U}}_i\cdot\underline{\underline{C}}\,\hat{\underline{U}}_i - \lambda_i^2\,\hat{\underline{U}}_i\cdot\hat{\underline{U}}_i = 0$$

$$\Rightarrow \quad \hat{\underline{u}}_i \cdot \underline{\underline{C}}\, \hat{\underline{u}}_i = \lambda_i^2$$

hence $\lambda(\hat{\underline{u}}_i)$ are principal stretches

ie extrema in stretch.

II Shear

The shear $\gamma(\hat{\underline{Y}}, \hat{\underline{Z}})$ at $\underline{X}$ is the change in angle between $\hat{\underline{Y}}$ and $\hat{\underline{Z}}$ going $\Omega_0 \to \Omega_t$

$$\boxed{\gamma(\hat{\underline{Y}}, \hat{\underline{Z}}) = \Theta(\hat{\underline{Y}}, \hat{\underline{Z}}) - \theta(\hat{\underline{Y}}, \hat{\underline{Z}})}$$

where

$$\lim_{\alpha \to 0} \cos\phi = \theta(\hat{\underline{Y}}, \hat{\underline{Z}})$$

To see this consider $\cos\phi = \dfrac{\underline{dy} \cdot \underline{dz}}{|\underline{dy}|\,|\underline{dz}|}$

where $\underline{dy} \cdot \underline{dz} = \underline{\underline{F}}\,\underline{dY} \cdot \underline{\underline{F}}\,\underline{dZ} = \underline{dY} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\,\underline{dZ}$

$$= \alpha^2\, \hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}$$

with $|\underline{dy}| = \alpha \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}$ $\qquad |\underline{dz}| = \alpha \sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\, \hat{\underline{Z}}}$

$$\Rightarrow \quad \cos\phi = \frac{\hat{\underline{Y}}\cdot \underline{\underline{C}}\,\hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}}\cdot \underline{\underline{C}}\,\hat{\underline{Y}}}\ \sqrt{\hat{\underline{Z}}\cdot \underline{\underline{C}}\,\hat{\underline{Z}}}} \xrightarrow{\alpha\to 0} \cos\theta(\hat{\underline{Y}},\hat{\underline{Z}})$$

Components of $\underline{\underline{C}}$

Let C_{IJ} be comp. of $\underline{\underline{C}}$ in $\{\underline{e}_I\}$

at any point $\underline{X}$ we have

$$\boxed{\begin{aligned} C_{II} &= \lambda^2(\underline{e}_I) \\ C_{IJ} &= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\,\sin\gamma(\underline{e}_I,\underline{e}_J) \end{aligned}}$$

$\Rightarrow$ diagonal components are square stretches in dir. of basis vectors

off-diagonal components are related to shears between the dir. of basis vectors.

The components $A_{ij} = \underline{e}_i \cdot \underline{\underline{A}}\,\underline{e}_j$

$$C_{II} = \underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_I = \lambda^2(\underline{e}_I) \checkmark$$

$$\cos\theta(\underline{e}_I, \underline{e}_J) = \frac{\underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_J}{\lambda(\underline{e}_I)\,\lambda(\underline{e}_J)} \qquad C_{IJ} = \underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_J$$

$$C_{IJ} = \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\cos\theta(\underline{e}_I, \underline{e}_J)$$

shear: $\gamma(\underline{e}_I, \underline{e}_J) = \underbrace{\Theta(\underline{e}_I, \underline{e}_J)}_{\frac{\pi}{2}} - \theta(\underline{e}_I, \underline{e}_J)$

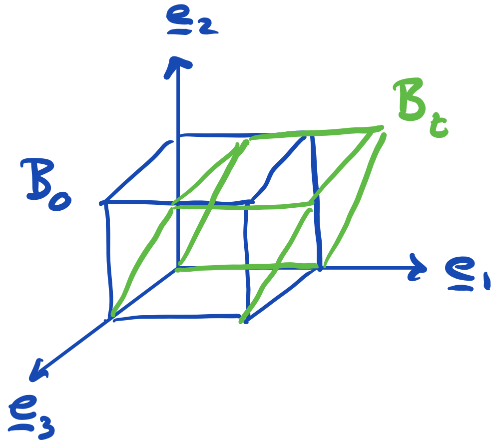
$$\theta(\underline{e}_I, \underline{e}_J) = \frac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)$$

substituting $\cos\left(\frac{\pi}{2} - \gamma\right) = \sin(\gamma)$

$$C_{IJ} = \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\,\sin(\gamma(\underline{e}_I, \underline{e}_J))$$

$\Rightarrow$ Components of $\underline{\underline{C}}$ quantify the strech of and the shear between basis vectors.

Example: Simple Shear



$$\underline{x} = \underline{\varphi}(\underline{X}) = \begin{bmatrix} X_1 + \alpha X_2 \\ X_2 \\ X_3 \end{bmatrix} \quad \alpha > 0$$

simple shear in $\underline{e}_1$-$\underline{e}_2$ plane

$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{hom. def.}$$

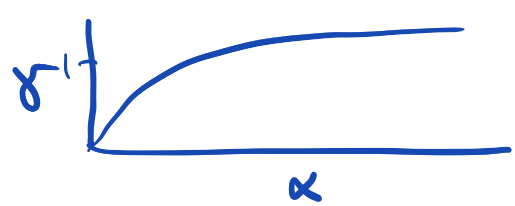
$$[\underline{\underline{C}}] = [\underline{\underline{F}}]^T[\underline{\underline{F}}] = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & 1+\alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find shear γ for $\underline{e}_1$ and $\underline{e}_2$

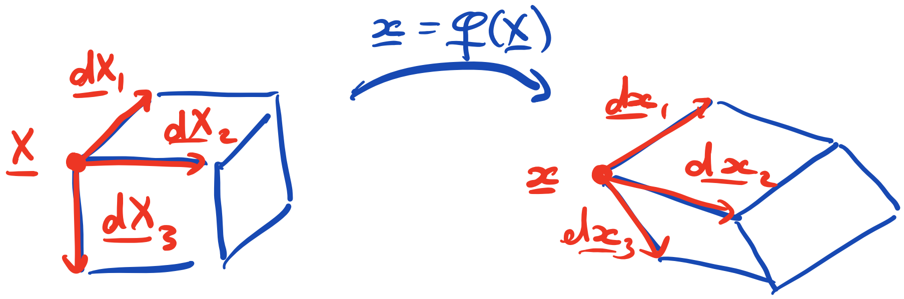
$$\gamma(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \theta(\underline{e}_1, \underline{e}_2)$$

$$\theta(\underline{e}_1, \underline{e}_2) = \frac{\underline{e}_1 \cdot \underline{\underline{C}}\, \underline{e}_2}{\lambda(\underline{e}_1)\, \lambda(\underline{e}_2)} = \frac{\alpha}{\sqrt{1}\,\sqrt{1+\alpha^2}} = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

$$\gamma(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - a\cos\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$$



Volume changes (from 3 lectures ago)



$$dV_X = (d\underline{X}_1 \times d\underline{X}_2) \cdot d\underline{X}_3 = \det(d\underline{\underline{X}})$$

$$d\underline{\underline{X}} = \left([d\underline{X}_1]\,[d\underline{X}_2]\,[d\underline{X}_3]\right)$$

$$dV_x = (d\underline{x}_1 \times d\underline{x}_2) \cdot d\underline{x}_3 =$$

$$d\underline{x}_i = \underline{\underline{F}}\, d\underline{X}_i$$

$$dV_x = \det\left([\underline{\underline{F}}\, d\underline{X}_1]\,[\underline{\underline{F}}\, d\underline{X}_2]\,[\underline{\underline{F}}\, d\underline{X}_3]\right)$$

$$= \det(\underline{\underline{F}}\, d\underline{\underline{X}}) = \det(\underline{\underline{F}}) \det(d\underline{\underline{X}})$$

$$\boxed{dV_x = \det(\underline{\underline{F}})\, dV_X}$$

The field $J(\underline{X}) = \det(\underline{\underline{F}}(\underline{X})) = \dfrac{dV_x}{dV_X}$ is the Jacobian of φ and it measures volume strain

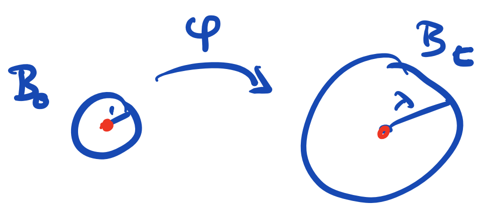
$J > 1$: volume increase

$J < 1$: volume decrease

$J = 1$: no volume change

admisee deformations must have $J > 0$

Exampe: Expanding shere $V = \frac{4}{3}\pi r^3$



$V_0 = \frac{4\pi}{3}$ $V_t = \frac{4\pi}{3}\lambda^3$

Deformation map: $\underline{x} = \varphi(\underline{X}) = \lambda \underline{X}$ $\lambda > 1$

$\underline{\underline{F}} = \nabla\varphi = \lambda \underline{\underline{I}}$ hom. $\Rightarrow J \neq J(\underline{X})$

$J = \det(\underline{\underline{F}}) = \det(\lambda I) = \lambda^3 \det(I) = \lambda^3$

$J = \frac{V_t}{V_0} = \lambda^3$

Homework question

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_{ij} = \underline{e}_i \cdot \underline{\underline{\sigma}}\, \underline{e}_j$

<u>uniaxial</u>

$\underline{\underline{\sigma}} = \sigma \quad \underline{a} \otimes \underline{a}$ $\qquad \underline{e}_1 = \underline{a}$

$\quad = \sigma \quad \underline{e}_1 \otimes \underline{e}_1$ $\qquad \underline{e}_2 \cdot \underline{e}_1 = 0$

$\underline{e}_3 \cdot \underline{e}_1 = 0$

$\sigma_{11} = \underline{e}_1 \left(\sigma \quad \underline{e}_1 \otimes \underline{e}_1\right) \underline{e}_1 = \sigma$

$\sigma_{12} = \qquad = 0$