Lecture 13: Infinitesimal strain

Logistics: - PS5 is due

- PS6 will be posted

Last time: - zoo of strain tensors

$\underline{\underline{C}} \quad \underline{\underline{b}} \quad \underline{\underline{E}} \quad \underline{\underline{e}}$

- Euler-Green strain relations

$\lambda(\hat{\underline{X}}) = \sqrt{\hat{\underline{X}} \cdot \underline{\underline{C}} \hat{\underline{X}}} \qquad \cos\theta(\hat{\underline{X}}, \hat{\underline{Y}}) = \dfrac{\hat{\underline{X}} \cdot \underline{\underline{C}} \hat{\underline{Y}}}{\lambda(\hat{\underline{X}}) \lambda(\hat{\underline{Y}})}$

- Components of $\underline{\underline{C}}$ $\qquad \gamma = \Theta - \theta$

$C_{II} = \lambda^2(\underline{e}_I) \quad C_{IJ} = \lambda(\underline{e}_I)\lambda(\underline{e}_J) \sin\gamma(\underline{e}_I, \underline{e}_J)$

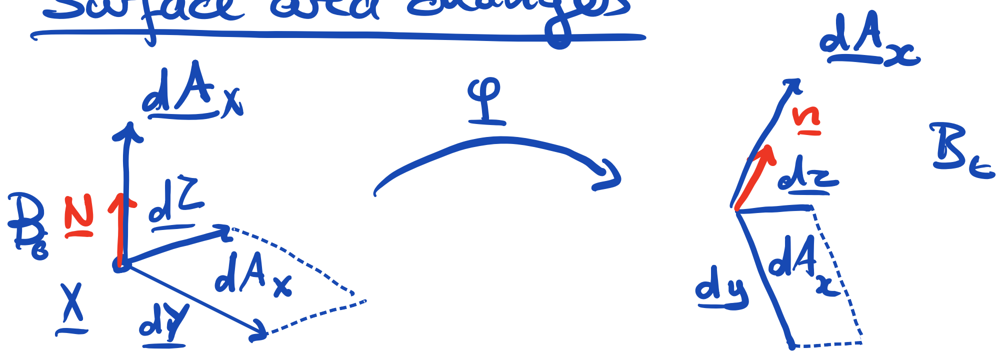
- Volume changes

$dV_x = J \, dV_X \qquad J = \det(\underline{\underline{F}})$

Today: - Changes in surface area

- Infinitesimal strain tensor

Surface area changes

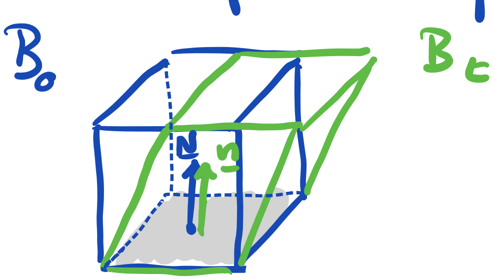


$dA_X = |\underline{dY} \times \underline{dZ}|$ $\qquad$ $dA_x = |\underline{dy} \times \underline{dz}|$

$\underline{dA}_X = \underline{dY} \times \underline{dZ} = \underline{N}\, dA_X \Big\}\ \underline{dA}_x = \underline{dy} \times \underline{dz} = \underline{n}\, dA_x$

$|\underline{n}| = |\underline{N}| = 1$ $\qquad$ $\underline{n} \neq \underline{\underline{F}}\,\underline{N}$!

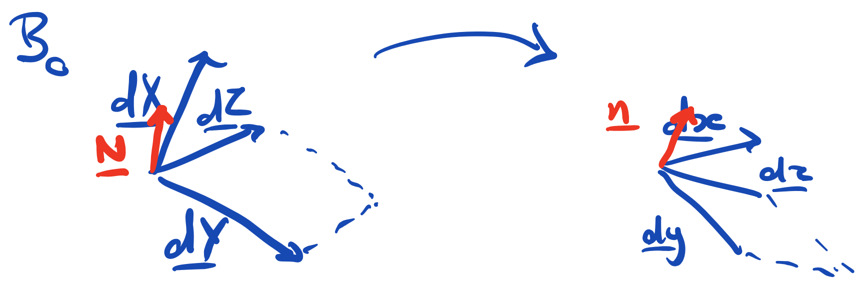
Example: simple shear



$\underline{N} = \underline{n} \neq \underline{\underline{F}}\,\underline{N}$

What is the relation between $\underline{n}$ and $\underline{N}$?
(in general?)

Consider $\underline{dX}$ so that $\underline{dX} \cdot \underline{N} \neq 0$



$$d\underline{A}_X = d\underline{Y} \times d\underline{Z} \qquad d\underline{a}_x = d\underline{y} \times d\underline{z}$$

$$dV_X = d\underline{A}_X \cdot d\underline{X} \qquad dv_x = d\underline{a}_x \cdot d\underline{x}$$

$$= (d\underline{Y} \times d\underline{Z}) \cdot d\underline{X}$$

Change in volume: $dv_x = J\, dV_X \qquad J = \det(\underline{\underline{F}})$

$$d\underline{a}_x \cdot d\underline{x} = J\, d\underline{A}_X \cdot d\underline{X} \qquad \text{with } d\underline{x} = \underline{\underline{F}}\, d\underline{X}$$

$$d\underline{a}_x \cdot \underline{\underline{F}}\, d\underline{X} = J\, d\underline{A}_X \cdot d\underline{X} \qquad \text{use transpose}$$

$$\underline{\underline{F}}^T d\underline{a}_x \cdot d\underline{X} = J\, d\underline{A}_X \cdot d\underline{X}$$

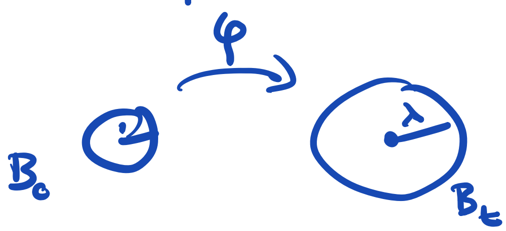
$$\underbrace{(\underline{\underline{F}}^T d\underline{a}_x - J\, d\underline{A}_X)}_{=0} \cdot d\underline{X} = 0 \qquad d\underline{X} \text{ is arbitrary}$$

$$\boxed{\begin{aligned} d\underline{a}_x &= J\, \underline{\underline{F}}^{-T} d\underline{A}_X \\ \underline{n}\, da_x &= J\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X \end{aligned}} \qquad \text{Nanson's formula}$$

$$d\underline{a}_x = \underline{n}\, da_x$$

so that $\quad \underline{n} = \underbrace{\frac{J\, dA_X}{dA_x}}_{\substack{\text{norm.} \\ |n|=1}} \; \underbrace{\underline{\underline{F}}^{-T}\underline{N}}_{\text{dir.}}$

Example: Expanding phere



$A_0 = 4\pi \qquad A_t = 4\pi\lambda^2$

$$\frac{A_t}{A_0} = \lambda^2$$

$$\underline{x} = \varphi(\underline{X}) = \lambda\,\underline{X} \qquad \underline{\underline{F}} = \lambda\,\underline{\underline{I}} \qquad J = \det(\underline{\underline{F}}) = \lambda^3$$

$$\underline{\underline{F}}^{-T} = \underline{\underline{F}}^{-1} = \frac{1}{\lambda}\,\underline{\underline{I}}$$

Nanson's formula: $\quad \underline{n}\, dA_x = J\,\underline{\underline{F}}^{-T}\underline{N}\, dA_X$

$$= \lambda^3 \frac{1}{\lambda}\underline{\underline{I}}\,\underline{N}\, dA_X$$

$$= \lambda^2 \underline{N}\, dA_X$$

$$\left|\underline{n}\frac{dA_x}{dA_X}\right| = \left|\lambda^2\,\underline{N}\right|$$

$$\underset{A_t}{dA_x} / \underset{A_0}{dA_X} = \lambda^2$$

Infinitesimal strain tensor

For any $\varphi : \mathcal{B}_0 \to \mathcal{B}_t$ with $\underline{u} = \varphi(\underline{X}) - \underline{X}$

we have displacement gradient $\nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}}$.

Another measure of strain is

$$\boxed{\underline{\underline{\varepsilon}} = \operatorname{sym}(\nabla \underline{u}) = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)}$$

$\underline{\underline{\varepsilon}}$ is the infinitesimal strain tensor

To relate $\nabla \underline{u}$ to $\underline{\underline{F}}$ and $\underline{\underline{C}}$

$$\nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}} \qquad \underline{\underline{F}} = \nabla \underline{u} + \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \operatorname{sym}(\underline{\underline{F}} - \underline{\underline{I}}) = \frac{1}{2}(\underline{\underline{F}} + \underline{\underline{F}}^T) - \underline{\underline{I}}$$

Given that $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ and $\underline{\underline{F}} = \nabla \underline{u} + \underline{\underline{I}}$

$$\underline{\underline{C}} = (\nabla \underline{u} + \underline{\underline{I}})^T (\nabla \underline{u} + \underline{\underline{I}}) = (\nabla \underline{u}^T + \underline{\underline{I}})(\nabla \underline{u} + \underline{\underline{I}})$$

$$= \nabla \underline{u}^T \nabla \underline{u} + \underbrace{\nabla \underline{u} + \nabla \underline{u}^T}_{2\underline{\underline{\varepsilon}}} + \underline{\underline{I}}$$

$$\boxed{\underline{\underline{\varepsilon}} = \underbrace{\tfrac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}_{\underline{\underline{E}}} - \tfrac{1}{2}\nabla\underline{u}^T\nabla\underline{u} = \underline{\underline{E}} - \tfrac{1}{2}\nabla\underline{u}^T\nabla\underline{u}}$$

$\underline{\underline{\varepsilon}}$ is useful for small deformations

$|\nabla\underline{u}| = O(\varepsilon)$ for $\underline{X}$ and $0 < \varepsilon \ll 1$.

$$\underline{\underline{\varepsilon}} = \underline{\underline{E}} + O(\varepsilon^2)$$

if terms of $O(\varepsilon^2)$ are neglected

$$\underline{\underline{\varepsilon}} = \underline{\underline{E}} = \tfrac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$$

Note : $\underline{\underline{\varepsilon}}$ is linear function of $\nabla\underline{u}$

$\underline{\underline{C}}$, $\underline{\underline{E}}$ are nonlinear functions.

$$\lim_{|\nabla u| \to 0} \underline{\underline{E}} = \underline{\underline{\varepsilon}}$$

$\Rightarrow$ important in linear elasticity.

Note: In the limit of infinitesimal deformation the distinction between reference and deformed configuration disappears.

Interpretation of components of $\underline{\underline{\varepsilon}}$

$$\boxed{\varepsilon_{ii} = \lambda(\underline{e}_i) - 1} \qquad \varepsilon_{ij} \approx \frac{1}{2} \sin \gamma(\underline{e}_i, \underline{e}_j)$$

$\lambda(\underline{e}_i)$ is strech in $\underline{e}_i$ dir

$\gamma(\underline{e}_i, \underline{e}_j)$ is shear between $\underline{e}_i$ and $\underline{e}_j$ dir

For diagonal components $\quad \underline{\underline{C}} = \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + \nabla u^T \nabla u$

$$C_{II} = 1 + 2\varepsilon_{ii} + \mathcal{O}(\varepsilon^2)$$

neglecting h.o.t $\quad C_{II} = 1 + 2\varepsilon_{ii}$

$$\sqrt{C_{II}} = \sqrt{1 + 2\varepsilon_{ii}} \qquad \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots \text{ Taylor}$$

$$= 1 + \varepsilon_{ii} \quad \Rightarrow \quad \varepsilon_{ii} = \sqrt{C_{II}} - 1 = \lambda(e_i) - 1 \checkmark$$

$$\lambda(\underline{e}_i) - 1 = \frac{|y - x| - |\underline{Y} - \underline{X}|}{|\underline{Y} - \underline{X}|} = \underbrace{\frac{|y - x|}{|\underline{Y} - \underline{X}|}}_{\lambda} - 1$$

$$\frac{\Delta L}{L} \Rightarrow \text{relative change in length}$$

$$\cos\theta(\underline{Y}, \underline{X}) = \frac{\hat{\underline{X}} \cdot \underline{\underline{C}}\, \hat{\underline{Y}}}{\lambda(\underline{X})\, \lambda(\underline{Y})}$$

For the off-diagonal components

$$\sin\gamma(\underline{e}_i, \underline{e}_j) = \frac{C_{ij}}{\sqrt{C_{ii}}\sqrt{C_{jj}}}$$

Last time: $C_{IJ} = \dfrac{\underbrace{\lambda(\underline{e}_i)}\,\lambda(\underline{e}_j)\sin\gamma(\underline{e}_I, \underline{e}_J)}{\sqrt{C_{II}}\sqrt{C_{JJ}}}$

solve for shear $I \to i$

$$\sin\gamma(\underline{e}_i, \underline{e}_j) = \frac{C_{ij}}{\sqrt{C_{ii}}\sqrt{C_{jj}}}$$

$$\underline{\underline{C}} = \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + O(\varepsilon^2)$$

$$C_{ij} = 2\varepsilon_{ij} + O(\varepsilon^2)$$

$$C_{ii} = 1 + O(\varepsilon)$$

$$\sqrt{C_{ii}}\sqrt{C_{jj}} = 1 + O(\varepsilon^2)$$

substituting into def. shear

$$e_{ij} = \frac{1}{2}\sin\gamma(\underline{e}_i, \underline{e}_j) \checkmark$$

if γ is small $\sin\gamma \to \gamma$

$$e_{ij} \approx \frac{1}{2}\gamma(\underline{e}_i, \underline{e}_j) = \frac{1}{2}\left(\Theta(\underline{e}_i, e_j) - \Theta(\underline{e}_i, \underline{e}_j)\right)$$

$\dot{H}$ Θ

Linearization of Kinematic Quantities

Given $\underline{x} = \underline{\varphi}(\underline{X})$ and $\underline{u} = \underline{x} - \underline{X}$

we have $\underline{\underline{H}} = \nabla \underline{u} = \underline{\underline{F}} - \underline{\underline{I}}$

what are the linearizations of

$\underline{\underline{U}}$ $\underline{\underline{V}}$ $\underline{\underline{R}}$ $\underline{\underline{C}}$ $\underline{\underline{E}}$

in the limit of $|\underline{\underline{H}}|$ small

$|\underline{\underline{H}}| = \sqrt{\underline{\underline{H}} : \underline{\underline{H}}} = \epsilon$

Using Taylor expansion it can be shown for any sym. tens. $\underline{\underline{A}}$ and $m \in \mathbb{R}$ that

$|\underline{\underline{A}}| = \epsilon$

$$(\underline{\underline{I}} + \underline{\underline{A}})^m = \underline{\underline{I}} + m\,\underline{\underline{A}} + \Theta(\epsilon^2) \quad \text{as}$$

using this we can show

$$\underline{\underline{C}} = \underline{\underline{U}}^2 = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T + O(\varepsilon^2)$$

$$\underline{\underline{b}} = \underline{\underline{V}}^2 = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{I}} + \underline{\underline{H}}^T + \underline{\underline{H}} + O(\varepsilon^2)$$

$$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + O(\varepsilon^2)$$

$$\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) + O(\varepsilon^2)$$

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{I}} + \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) + O(\varepsilon^2)$$

identify two tensors

$$\boxed{\begin{aligned} \underline{\underline{\varepsilon}} &= \tfrac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) = \mathrm{sym}(\underline{\underline{H}}) \\ \underline{\underline{\omega}} &= \tfrac{1}{2}(\underline{\underline{H}} - \underline{\underline{H}}^T) = \mathrm{skew}(\underline{\underline{H}}) \end{aligned}}$$

inf. strain/stretch

inf. rotation

Decomposition into stretch & rotation

$$\underline{\underline{F}} = \underline{\underline{H}} + \underline{\underline{I}} = \underline{\underline{I}} + \mathrm{sym}(\underline{\underline{H}}) + \mathrm{skew}(\underline{\underline{H}})$$

$$= \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$$

$\Rightarrow$ rotation & stretch are additive

Finite deformation:

$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$ multiplicative

$$\underline{\underline{F}} = (\underline{\underline{I}} + \underline{\underline{\omega}} + O(\varepsilon^2))(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + O(\varepsilon^2))$$

$$= \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}} + O(\varepsilon^2)$$