Lecture 14: Motions & Material time derivative

Logistics: - PS6 has been updated

make use of office hrs tomorrow 3-3:30pm

Last time: - Nansen's Formula $\underline{n}\, dA_x = J \underline{\underline{F}}^{-T} \underline{N}\, dA_X$

- Infinitesimal strain tensor

$$\underline{u} = \varphi(\underline{X}) - \underline{X} \qquad \nabla \underline{u} = \underline{\underline{H}} = \underline{\underline{F}} - \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = \operatorname{sym}(\nabla \underline{u}) = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\underline{\underline{\varepsilon}} = \underbrace{\frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})}_{\underline{\underline{E}}} - \frac{1}{2}\nabla u^T \nabla u$$

$$\lim_{|\nabla u| \to 0} \underline{\underline{E}} = \underline{\underline{\varepsilon}}$$

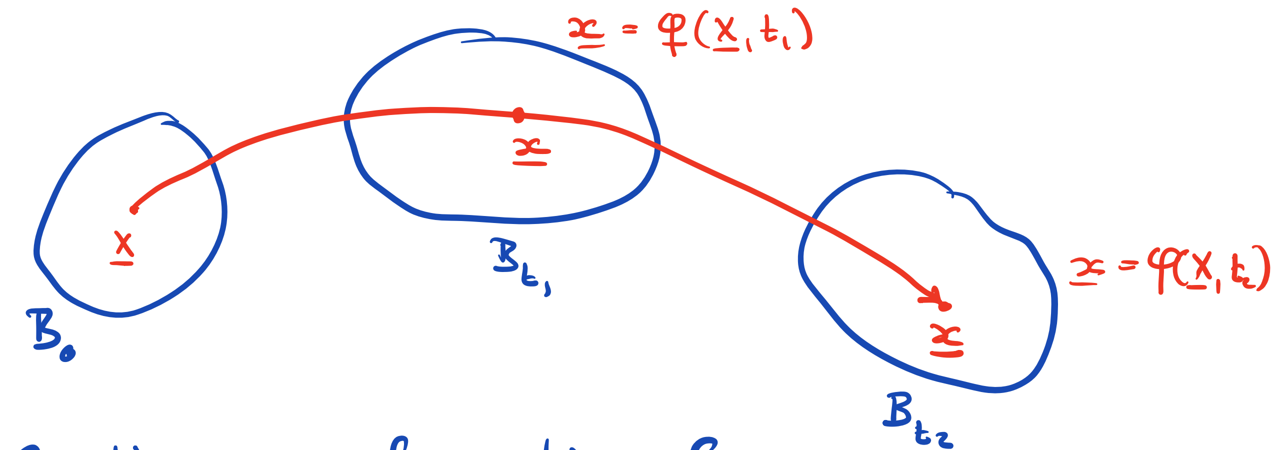
$$\varepsilon_{ii} = \lambda(\underline{e}_i) - 1 \qquad \varepsilon_{ij} = \frac{1}{2}\sin\gamma(\underline{e}_i, \underline{e}_j)$$

Today: - Motions

- Spatial & Material fields

- Time derivatives

Motions



Continuous deformation of body over time is called a motion, $\varphi(\underline{X}, t)$

$$\varphi_t(\underline{X}) = \varphi(\underline{X}, t)$$

Inverse motion : $\underline{\psi}(\underline{x}, t) = \varphi^{-1}(\underline{X}, t) =$

$$= \underline{\psi}_t(\underline{x})$$

Assume both φ and ψ are smooth.

Material & Spatial Fields

some field naturally defined on current configuration $T(\underline{x}, t)$

others naturally defined in reference config.

eg grain size, density

φ and ψ allow us to any field as function of both $\underline{X}$ and $\underline{x}$.

Material field is a field expressed in terms of $\underline{X} \in B_0$ $\quad \Omega = \Omega(\underline{X}, t)$

Spatial field is a field expressed in terms of $\underline{x} \in B_t$ $\quad \Gamma = \Gamma(\underline{x}, t)$

$$\underline{x} = \varphi(\underline{X}, t) \qquad \underline{X} = \psi(\underline{x}, t)$$

To any material field we can associate a spatial field

$$\Omega_s(\underline{x}, t) = \Omega(\psi(\underline{x}, t), t)$$

and Ω_s is spatial description of Ω

To any spatial field Γ we can associate a material field

$$\Gamma_m(\underline{X}, t) = \Gamma(\varphi(\underline{X}, t), t)$$

we call Γ_m the material description of Γ.

Coordinate derivatives:

Material coord: ∇_X = Grad, Div, Curl, Lap

Spatial coord: ∇_x = grad, div, curl, lap

Velocity and Acceleration Fields

The velocity and acceleration of material particle labled $X \in B_0$ at time t due to motion $\varphi(\underline{X}, t)$ are given by

$$\boxed{\begin{aligned} \underline{V}(\underline{X}, t) &= \frac{\partial}{\partial t}\varphi(\underline{X}, t) = \left.\frac{\partial \underline{x}}{\partial t}\right|_{\underline{X}} \\ \underline{\underline{A}}(\underline{X}, t) &= \frac{\partial^2}{\partial t^2}\varphi(\underline{X}, t) = \left.\frac{\partial^2 \underline{x}}{\partial t^2}\right|_{\underline{X}} \end{aligned}}$$

vel. and acc. are naturally material field because they are associated with a particle.

The spatial description of of these fields are

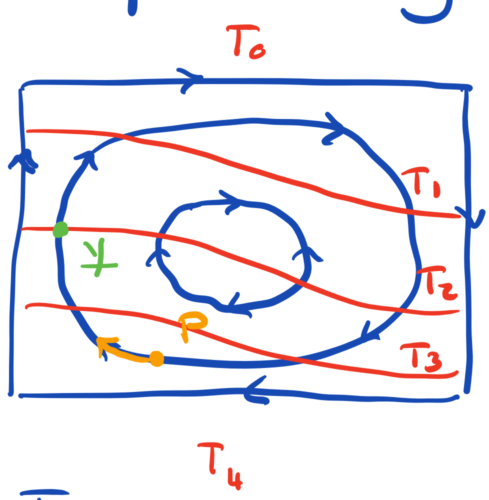
$$\underline{v}(\underline{x},t) = V_s(\underline{x},t) = \frac{\partial}{\partial t}\varphi(\psi(\underline{x},t),t)$$

$$\underline{a}(\underline{x},t) = A_s(\underline{x},t) = \frac{\partial^2}{\partial t^2}\varphi(\psi(\underline{x},t),t)$$

The spatial fields correspond to the material particle at $\underline{x}$ at t.

Note: $\underline{a} \neq \frac{\partial v}{\partial t}$

Example: Steady Convection

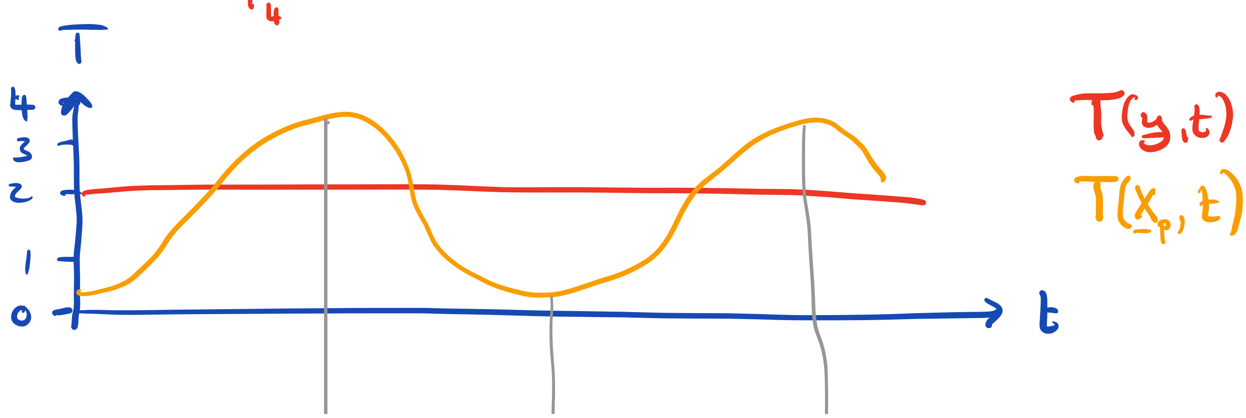


$T_0 < T_1 < T_2 < T_3 < T_4$

– T contours

– Streamlines

Steady state: $T(\underline{x},t) = T(\underline{x})$



- local time derivative at y is zero, because spatial field is steady
- ~~the~~ Time deriv. following particle is oscillating as particle goes round.
- Time derivative of successive particles passing by ~~y~~

Different time derivatives

I) Material time derivative of material field Ω

Deriv. of Ω with t holding $\underline{X}$ fixed

$$\boxed{\dot{\Omega}(\underline{X}, t) = \frac{D\Omega}{Dt}(\underline{X}, t) = \frac{\partial \Omega}{\partial t}\Big|_{\underline{X}}}$$

also called total, substantial, convective deriv.

$\dot{\Omega}$ represents the rate of change of Ω seen by an observer following the pathline of a particle.

II Spatial time derivative of a spatial field

Derivative Γ with respect et t holding x fixed

$$\frac{\partial \Gamma}{\partial t}(\underline{x},t)\Big|_{\underline{x}} = \frac{\partial \Gamma}{\partial t}(\underline{x},t) \qquad \text{———} = 0$$

referred to as local time derivative

$\frac{\partial \Gamma}{\partial t}$ is rate of change of Γ seen by an observer located at $\underline{x}$.

III Material time derivative of spatial field

Derivative of spatial field Γ with respect to time holding $\underline{X}$ fixed.

$$\dot{\Gamma}(\underline{x},t) = \frac{D\Gamma}{Dt}(\underline{x},t) = \frac{\partial}{\partial t} \underbrace{\Gamma(\varphi(\underline{X},t),t)}_{\Gamma_m(\underline{X},t)}\Big|_{\underline{X}=\varphi^{-1}(\underline{x},t)}$$

two time dependencies !

By chain rule

$$\frac{\partial}{\partial t}\Gamma(\underbrace{\varphi(\cancel{\underline{X}},t)}_{\underline{x}},t) = \frac{\partial\Gamma}{\partial t}\Big|_{\underline{x}=\varphi(\cancel{\underline{X}},t)} + \frac{\partial\Gamma}{\partial x_i}\Big|_{x=\varphi(\underline{X},t)} \underbrace{\frac{\partial x_i}{\partial t}}_{\frac{\partial\varphi_i}{\partial t}=v_i}$$

$$= \frac{\partial\Gamma}{\partial t}\Big|_{x=\varphi(\underline{X},t)} + \frac{\partial\Gamma}{\partial x_i}\Big|_{\underline{x}=\varphi(\underline{X},t)} v_i(\underline{x},t)\Big|_{\underline{x}=\varphi(\underline{X},t)}$$

$$\dot{\Gamma}(\varphi(X,t),t) = \left[\frac{\partial\Gamma}{\partial t}(\underline{x},t) + \frac{\partial\Gamma}{\partial x_i}(\underline{x},t)\, v_i(\underline{x},t)\right]_{\underline{x}=\varphi(\underline{X},t)}$$

material derivative of spatial field Γ in material coordinates.

Expressing the result in spatial coord.

$$\dot{\Gamma}(\underline{x},t) = \frac{\partial\Gamma}{\partial t}(\underline{x},t) + \frac{\partial\Gamma}{\partial x_i}(\underline{x},t)\, v_i(\underline{x},t)$$

Let $\varphi(\underline{X},t)$ be a motion with spatial velocity field $\underline{v}(\underline{x},t)$ then consider scalar field $\phi(\underline{x},t)$ and vector field $\underline{w}(\underline{x},t)$ then the spatial representation of their material time derivatives are given

$$\boxed{\dot{\phi} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla_x \phi}$$ and $$\boxed{\dot{\underline{\omega}} = \frac{\partial \underline{\omega}}{\partial t} + (\nabla_x \underline{v})\, \underline{\omega}}$$

$$(\underline{v} \cdot \nabla)\, \underline{\omega}$$

result for $\underline{\omega}$ follows by applying scalar result to ω_i

Important because we can compute $\dot{\phi}$ and $\dot{\underline{\omega}}$ without knowledge of motion.
$\Rightarrow$ fluid mechanics we never see $\underline{\varphi}$

Most important application: spatial acceleration

$$\boxed{\underline{a} = \dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\, \underline{v}}$$

For example $\Rightarrow$ Lecture notes on motions