Lecture 15: Rate of deformation & Reynolds Transp. Thm

Logistics: - no HW, my apologies

Last time: - Motions $\varphi(\underline{X},t)$ & $\psi(\underline{x},t) = \varphi^{-1}(\underline{x},t)$

- spatial & material fields $\Gamma(\underline{x},t)$ $\Omega(\underline{X},t)$

  spatial and material representations

  $\Gamma_m(\underline{X},t) = \Gamma(\varphi(\underline{X},t),t)$, $\Omega_s(\underline{x},t) = \Omega(\psi(\underline{x},t),t)$

- Spatial & material derivatives

  $\dot{\Omega}(\underline{X},t) = \frac{\partial}{\partial t}\Omega(\underline{X},t)$

  $$\boxed{\dot{\Gamma}_s(\underline{x},t) = \frac{\partial \Gamma}{\partial t} + \underline{v}\cdot\nabla\Gamma}$$

  spatial representation of material derivative of a spatial field!

  $\Rightarrow$ independent of φ!

Today: More on rates

- Rate of strain & spin tensors
- Reynolds transport theorem

Rate of deformation tensors

Velocity gradient takes role of def. gradient

Spatial velocity gradient:

$$\underline{\underline{l}} = \nabla_x \underline{v} \qquad l_{ij} = \frac{\partial v_i}{\partial x_j}$$

Material velocity gradient:

$$\underline{\underline{F}} = \nabla_X \varphi \qquad F_{iJ} = \varphi_{i,J}$$

$$\underline{V} = \dot{\varphi} \qquad V_i = \varphi_{i,t}$$

$$\dot{\underline{\underline{F}}} = \frac{\partial}{\partial t}(\nabla_X \varphi) = \nabla_X\left(\frac{\partial \varphi}{\partial t}\right) = \nabla_X \underline{V} \qquad \dot{F}_{iJ} = V_{i,J}$$

Deformation:
$$\varphi(\underline{X} + \Delta\underline{X}) = \varphi(\underline{X}) + \underline{\underline{F}}\,\Delta\underline{X}$$

$$\underline{V}(\underline{X} + \Delta\underline{X}) = \underline{V}(\underline{X}) + \dot{\underline{\underline{F}}}\,\Delta\underline{X}$$

$$= \underline{V}(\underline{X}) + \nabla_X \underline{V}\,\Delta\underline{X}$$

Note: $\underline{V}(\underline{X}, t) = \underline{v}(\underbrace{\varphi(\underline{X}, t)}_{\underline{x}}, t)$

$\underline{V}$ and $\underline{v}$ are same vector field just expressed in a different variable

$$\nabla_X \underline{\underline{V}} \neq \nabla_x \underline{\underline{v}} \big|_{x = \varphi(\underline{X}, t)}$$

because gradients are in different directions

$$\dot{F}_{iJ} = \frac{\partial}{\partial X_J} V_i = \frac{\partial}{\partial X_J} v_i(\varphi(\underline{X}, t), t)$$

where $\frac{\partial}{\partial X_J} = \frac{\partial}{\partial x_k} \frac{\partial x_k}{\partial X_J} = \frac{\partial}{\partial x_k} \varphi_{k,J} = \frac{\partial}{\partial x_k} F_{kJ}$

substitute

$$\dot{F}_{iJ} = \frac{\partial}{\partial x_k} v_i(\underline{x}, t) \; F_{kJ} = v_{i,k} F_{kJ}$$

$$\dot{\underline{\underline{F}}} = \nabla_X \underline{\underline{V}} = \nabla_x \underline{\underline{v}} \; \underline{\underline{F}}$$

$$\Rightarrow \boxed{\underline{\underline{\ell}} = \nabla_x \underline{\underline{v}} = \dot{\underline{\underline{F}}} \, \underline{\underline{F}}^{-1}}$$

Need to decompose $\underline{\underline{\ell}}$ just like $\underline{\underline{F}}$

Finite strain $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$ $\quad \nabla \varphi \quad$ $\underline{u} = \varphi - \underline{X}$

infinitesimal strain : $\nabla \underline{u} = \mathrm{sym}(\nabla u) + \mathrm{skew}(\nabla u)$

Decomposition of $\underline{\underline{\ell}}$

Because we are interested in instantaneous rates of change we can use additive decomp.

$$\boxed{\begin{aligned} &\underline{\underline{\ell}}(\underline{x},t) = \nabla_x \underline{v} = \underline{\underline{d}} + \underline{\underline{w}} \\ &\underline{\underline{d}} = \text{sym}(\nabla_x \underline{v}) = \tfrac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T) \\ &\underline{\underline{w}} = \text{skew}(\nabla_x \underline{v}) = -\tfrac{1}{2}(\nabla_x \underline{v} - \nabla_x \underline{v}^T) \end{aligned}}$$

rate of strain

spin tensor

Interpretation of $\underline{\underline{d}}$ and $\underline{\underline{\ell}}$

$\underline{v}(\underline{x} + \underline{\Delta x}) \approx \underline{v}(\underline{x}) + \nabla_x \underline{v}\, \underline{\Delta x}$ $\qquad \nabla_x \underline{v} = \underline{\underline{d}} + \underline{\underline{w}}$

$\approx \underline{v}(\underline{x}) + \underline{\underline{d}}\, \underline{\Delta x} + \underline{\underline{w}}\, \underline{\Delta x}$

$\underline{\underline{w}} = -\underline{\underline{w}}^T \rightarrow$ axial vector $\quad \underline{\underline{w}}\, \underline{\Delta x} = \underline{\omega} \times \underline{\Delta x}$

$\approx \underline{v}(\underline{x}) + \underline{\underline{d}}\, \underline{\Delta x} + \underline{\omega} \times \underline{\Delta x}$

$\underline{\underline{d}}$ is rate of change in shape (strain rate)

$\underline{\underline{w}}$ is rate of change in orientation (spin)

$\underline{\omega}$ is angular velocity

vorticity: $\nabla_x \times \underline{v} = 2\underline{\omega}$ $\qquad 2 \times$ spin

Relation of $\underline{\underline{l}}$ to $\underline{\underline{U}}, \dot{\underline{\underline{U}}}, \underline{\underline{R}}, \dot{\underline{\underline{R}}}$ $(\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}})$

$$\underline{\underline{l}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$$

Material derivatives: $\dot{\underline{\underline{F}}}^{-1}$ $\quad \dot{\underline{\underline{F}}}^{-T}$

$$\dot{\underline{\underline{I}}} = \frac{d}{dt}(\underline{\underline{F}}^{-1}\underline{\underline{F}}) = \dot{\underline{\underline{F}}}^{-1}\underline{\underline{F}} + \underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}} = \underline{\underline{0}}$$

$$\dot{\underline{\underline{F}}}^{-1} = -\underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = -\underline{\underline{F}}^{-1}\underline{\underline{l}}$$

$$\dot{\underline{\underline{F}}}^{-T} = (\dot{\underline{\underline{F}}}^{-1})^{T} = -(\underline{\underline{F}}^{-1}\underline{\underline{l}})^{T} = -\underline{\underline{l}}^{T}\underline{\underline{F}}^{-T}$$

$$\boxed{\dot{\underline{\underline{F}}}^{-1} = -\underline{\underline{F}}^{-1}\underline{\underline{l}}} \qquad \boxed{\dot{\underline{\underline{F}}}^{-T} = -\underline{\underline{l}}^{T}\underline{\underline{F}}^{-T}}$$

substitute $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$ into $\underline{\underline{l}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$$\underline{\underline{l}} = \frac{d}{dt}(\underline{\underline{R}}\,\underline{\underline{U}})\,\underline{\underline{F}}^{-1} = (\dot{\underline{\underline{R}}}\,\underline{\underline{U}} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}})(\underline{\underline{R}}\,\underline{\underline{U}})^{-1}$$

$$= (\qquad '' \qquad)\,\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T}$$

$$= \dot{\underline{\underline{R}}}\,\underbrace{\underline{\underline{U}}\,\underline{\underline{U}}^{-1}}_{\underline{\underline{I}}}\,\underline{\underline{R}}^{T} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T}$$

$$\underline{\underline{l}} = \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^{T} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T}$$

show $\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^{T} = -(\dot{\underline{\underline{R}}}\,\underline{\underline{R}})^{T}$ $\qquad \frac{d}{dt}(\underline{\underline{R}}^{T}\underline{\underline{R}}) = \underline{\underline{0}}$

second term is in general not symmetric

$\underline{\underline{l}} = \underline{\underline{d}} + \underline{\underline{w}}$

$\underline{\underline{d}} = \mathrm{sym}(R\dot{U}U^{-1}R^T) = \underline{\underline{R}}\ \mathrm{sym}(\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1})\,R^T$

$\underline{\underline{w}} = \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T + R\ \mathrm{skew}(\dot{\underline{U}}U^{-1})\,R^T$

Shows that $\underline{\underline{d}}$ is not a pure rate of strain

and $\underline{\underline{w}}$ is not a pure rate of rotation

It remains to be seen why we can use

$\underline{\underline{d}}$ in shear viscosity.

Reynolds Transport Thm

motion $\varphi(\underline{X},t)$ with $\underline{v}(\underline{x},t)$ and

$\Omega_t \subset B_t$ with surface $\partial\Omega_t$ and

outward unit normal $\underline{n}$.

$$\boxed{\frac{d}{dt}\int_{\Omega_t}\phi(\underline{x},t)\,dV_x = \underbrace{\int_{\Omega_t}\frac{\partial\phi}{\partial t}\,dV_x}_{\text{change in }\Omega} + \underbrace{\oint_{\partial\Omega_t}\phi\,\underline{v}\cdot\underline{n}\,dA_x}_{\substack{\text{flux across }\partial\Omega_t\\ \text{due motion of }\Omega.}}}$$

Key: Although $\Omega_t = \varphi(\Omega_0,t)$ we can compute derivative without knowledge of φ

Move to reference config.

$$\frac{d}{dt}\int_{\Omega_t}\phi(\underline{x},t)\,dV_x = \frac{d}{dt}\int_{\Omega_0}\underbrace{\phi(\varphi(\underline{X},t),t)}_{\phi_m(X,t)}\,J(\underline{X},t)\,dV_X$$

Ω_0 is fixed exchange deriv. & integral

$$\int_{\Omega_0}\frac{d}{dt}(\phi_m J)\,dV_X = \int_{\Omega_0}(\dot{\phi}_m J + \phi_m \dot{J})\,dV_X$$

where $\dot{J} = J\,(\nabla_x \cdot \underline{v})_m$ $\longrightarrow$ show later

$$= \int_{\Omega_0} \dot{\phi}_m J + \phi_m J\,(\nabla_x \cdot \underline{v})_m \; dV_X$$

$$= \int_{\Omega_0} \dot{\phi}_m + \phi_m (\nabla_x \cdot \underline{v})_m \; \underbrace{J\,dV_X}_{dV_x}$$

$$= \int_{\Omega_t} \dot{\phi} + \phi \; \nabla_x \cdot \underline{v} \;\; dV_x$$

subst. spatial descrip. of material deriv

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla_x \phi$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \nabla_x \cdot (\phi \underline{v}) \; dV_x$$

divergence thm

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} \, dV_x + \oint_{\partial \Omega} \phi \underline{v} \cdot \underline{n} \; dA_x \; \checkmark$$

What about $\dot{J} = J\,(\nabla_x \cdot \underline{v})_m$? $\quad J = \det(\underline{F})$

From lecture 5:

Deriv. of a scalar valued tensor fun:

$$\dot{\psi}(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}$$

Deriv. of determinant: $D\det(\underline{\underline{S}}) = \det(\underline{\underline{S}})\,\underline{\underline{S}}^{-T}$

From lecture 3: $\underline{\underline{S}} : \underline{\underline{D}} = tr(\underline{\underline{S}}^T \underline{\underline{D}})$

$$\Rightarrow \dot{J} = \frac{d}{dt}\det(\underline{\underline{F}}) = \det(\underline{\underline{F}})\,\underline{\underline{F}}^{-T} : \dot{\underline{\underline{F}}} = J\, tr(\underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}})$$

$$= J\, tr(\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1})$$

using: $\nabla_x v = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$$\dot{J} = J\, tr(\nabla_x \underline{v}) = J\,(\nabla_x \cdot \underline{v})_m$$