Lecture 16: Integral balance laws

Logistics: - no HW

- try to post new HW

Last time: - Velocity gradients

spatial: $\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

material: $\dot{\underline{\underline{F}}} = \nabla_X \underline{V}$

$\Big\}\ \nabla_X \underline{V} = \nabla_x \underline{v}\, \underline{\underline{F}}$

- $\underline{\underline{\ell}} = \underline{\underline{d}} + \underline{\underline{w}}$

$\underline{\underline{d}} = \mathrm{sym}(\underline{\underline{\ell}}) = \frac{1}{2}\left(\nabla_x \underline{v} + \nabla_x \underline{v}^T\right)$ rate of strain

$\underline{\underline{w}} = \mathrm{skew}(\ell) = \frac{1}{2}\left(\nabla_x \underline{v} + \nabla_x \underline{v}^T\right)$ spin

- Reynolds Transp. Thm

$$\frac{d}{dt}\int_{\Omega_t} \phi \, dV_x = \int_{\Omega_t} \frac{\partial \phi}{\partial t}\, dV_x + \oint_{\partial\Omega_t} \phi \underline{v}\cdot\underline{n}\, dV_x$$

$\Rightarrow$ we don't need to know φ

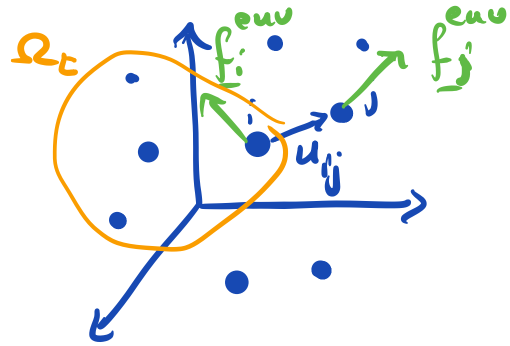
Today: - Balance laws

Discrete $\rightarrow$ continuum

- Balance laws in integral form

- Continuum Thermo

Discrete system



N particles

m_i = masses

$\underline{x}_i$ = locations

$u_{ij} = u_{ji}$ interaction energy

$\underline{f}_{ij}^{int} = -\nabla_x u_{ij}$ interaction force

$\underline{f}_i^{env}$: environ. forces

Mass conservation: $\dot{m}_i = 0$

Newtons 2^{nd} law: $m_i \ddot{\underline{x}}_i = \underline{f}_i^{env} + \sum_{i \neq j} \underline{f}_{ij}^{int}$

For any subset Ω_t of particles: $i \in I \subset [1,..N]$

total mass: $M[\Omega_t] = \sum_{i \in I} m_i$

lin mom.: $\underline{L}[\Omega_t] = \sum_{i \in I} m_i \dot{\underline{x}}_i$

ang. mom.: $\underline{j}[\Omega_t] = \sum_{i \in I} \underline{x}_i \times \underline{m}_i \dot{\underline{x}}_i$

internal energy: $U[\Omega_t] = \sum_{i \in I} u_{ij}$

kinetic energy: $K[\Omega_t] = \sum_{i\in I} \frac{1}{2} m_i |\dot{\underline{x}}|^2$

We have the following discrete balance laws:

Mass is conserved: $\frac{d}{dt} M[\Omega_t] = 0$

Change in lin. and ang. mom are equal to resultant ext. force and torque on Ω_t.

$$\frac{d}{dt}\underline{l}[\Omega_t] = \sum_{i\in I}\Big[\underline{f}_i^{env} + \sum_{j\notin I} f_{ij}^{int}\Big]$$

$$\frac{d}{dt}\underline{j}[\Omega_t] = \sum_{i\in I}\underline{x}_i \times \Big[\underline{f}_i^{env} + \sum_{j\notin I}\underline{f}_{ij}^{int}\Big]$$

Change in internal and kinetic energy is due to the power of external forces

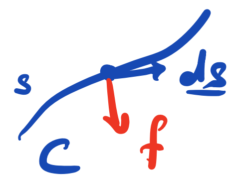
$$\frac{d}{dt}\big(U[\Omega_t] + K[\Omega_t]\big) = \sum_{i\in I}\underbrace{\dot{\underline{x}}}_{\underline{v}} \cdot \Big[\underbrace{\underline{f}_i^{env} + \sum_{j\notin I}\underline{f}_{ij}^{int}}_{\underline{f}}\Big]$$

Reminder:

Work is energy transferred by application of a force along a distance.

$$W = Fs$$

$$W = \int_C \underline{f} \cdot d\underline{s} = \int_{t_0}^{t_1} \underline{f} \cdot \underbrace{\frac{d\underline{s}}{dt}}_{\underline{v}} \, dt$$



$$= \int \underline{f} \cdot \underline{v} \, dt = \int \frac{dW}{dt} \, dt$$

$$\Rightarrow \frac{dW}{dt} = \underline{f} \cdot \underline{v}$$

Power: $\boxed{P = \frac{dW}{dt} = \underline{v} \cdot \underline{f}}$

To generalize balance laws to continuum

$\Sigma \rightarrow \int$ for mass, lin & ang mom.

but continuum energy balance is more complicated.

continuum velocity is mean velocity

$\Rightarrow$ loose information abut velocity fluctuations

Introduce new variables:

Temperature :- measure of the magnitude of the velocity fluctuations

Heat : measure of the energy in fluctuations

Balance laws in integral form

continuum def. mass, lin. & ang. mom.

of $\Omega_t \in B_t$: $M[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_x$

$$\underline{L}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, \underline{v}(\underline{x},t)\, dV_x$$

$$\underline{j}[\Omega_t] = \int_{\Omega_t} (\underline{x}-\underline{z}) \times \rho(\underline{x},t)\, \underline{v}(\underline{x},t)\, dV_x$$

Conservation of mass:

In absence of relativistic effects or radio active decay the mass of body

~~obs~~ does not change as it deforms:

$$\boxed{\frac{d}{dt} M[\Omega_t] = 0} \quad \text{for all } \Omega_t \subset B_t$$

Laws of inertia

In fixed reference frame

$$\frac{d}{dt} \underline{l}[\Omega_t] = \int_{\Omega_t} \rho(x,t)\, \underline{b}(\underline{x},t)\, dV_x + \int_{\partial\Omega} \underline{t}(\underline{x},t)\, dA_x$$

$$\frac{d}{dt} \underline{j}[\Omega_t] = \int_{\Omega_t} \underline{x} \times \rho(\underline{x},t) b(\underline{x},t)\, dV + \int_{\partial\Omega} \underline{x} \times t(\underline{x},t)\, dA_x$$

Continuum Thermodynamics

Assume existence of abs. temperature field $\Theta(\underline{x},t) \geq 0$ at all $\underline{x} \in B_t$. It is a measure of the velocity fluctuations of atoms in vicinity of $\underline{x}$.

Thermal energy or heat content is the energy associated with temp./fluctuations.

Bodies can gain/lose heat in two ways:

I) Body heating: $Q_b[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, r(\underline{x},t)\, dV_x$

II) Surface heating: $Q_s[\Omega_t] = -\int_{\partial\Omega_t} q(\underline{x},t)\cdot n\, dA_x$

where $r(x,t)$ is heat supply/loss per unit mass.

and $q(\underline{x},t)$ is the heat flux vector.

Net heating

$$\dot{Q}[\Omega_t] = \dot{Q}_b[\Omega_t] + \dot{Q}_s[\Omega_t] = \int_{\Omega_t} \rho r\, dV_x - \int_{\partial\Omega_t} q\cdot n\, dA_x$$

Kinetic Energy: $K[\Omega_t] = \int_{\Omega_t} \frac{1}{2}\rho\, |\underline{v}|^2\, dV_x$

Power of external forces:

$$P[\Omega_t] = \int_{\Omega} \rho\, \underline{b}\cdot\underline{v}\, dV_x + \int_{\partial\Omega} \underline{t}\cdot\underline{v}\, dA_x$$

Net working $W[\Omega_t]$ of external forces on Ω_t is the mech. power that is not converted into motion.

$$\dot{W}[\Omega_t] = P[\Omega_t] - \frac{d}{dt} K[\Omega_t]$$

$W[\Omega_t] > 0$: mechanical energy is stored

$W[\Omega_t] < 0$: mechanical energy is released

Internal energy & the first law

Energy content of body not assoc. with kinetic energy is called the internal energy.

Here we assume thermal, mechanical (elastic) internal energy is

$$U[\Omega_t] = \int_{\Omega_t} \rho \phi \, dV_x$$

ϕ is internal energy density per unit mass

First law of Thermo

$$\boxed{\frac{d}{dt} U[\Omega_t] = Q[\Omega_t] + W[\Omega_t]}$$

or

$$\frac{d}{dt}\left(U[\Omega_t] + K[\Omega_t]\right) = Q[\Omega_t] + P[\Omega_t]$$

Note: Discrete energy balance does <u>not</u> have the net heating term

In some cases, the power of an external force can be written as $P[\Omega_t] = -\frac{d}{dt} G[\Omega_t]$ where $G[\Omega_t]$ is the potential energy for the external force

$$\frac{d}{dt}\Big(\underset{\substack{\uparrow \\ \text{internal}}}{U[\Omega_t]} + \underset{\substack{\uparrow \\ \text{kinetic}}}{K[\Omega_t]} + \underset{\substack{\uparrow \\ \text{potential}}}{G[\Omega_t]}\Big) = Q[\Omega_t] \leftarrow \underset{\uparrow}{P_L}$$

<u>Entropy and 2^{nd} Law</u>

The second law expresses the fact that a body has ~~to~~ a limit on the rate of heat uptake but no limit on rate of heat release.

$$Q[\Omega_t] \leq \Xi[\Omega_t]$$

where $\Xi[\Omega_t]$ is upper bound on the net heating.

In absence of net working $W[\Omega_t] = 0$

$$\frac{dU}{dt}[\Omega_t] = Q[\Omega_t] \leq \Xi[\Omega_t]$$

rate of energy storage is limited

Entropy of a body is defined

$$\boxed{\frac{d}{dt} S[\Omega_t] = \frac{\Xi[\Omega_t]}{\Theta[\Omega_t]}}$$

Θ is mean temp.

entropy is the quantity whose rate of change is the upper bound heating per unit temperature.

In terms of net heating

$$\boxed{\frac{d}{dt} S[\Omega_t] \geq \frac{Q[\Omega_t]}{\Theta[\Omega_t]}}$$

Clausius-Planck inequality

In thermo books: $dS = \frac{dQ_{rev}}{T}$ $\quad dS \geq \frac{dQ}{T}$

For a non-hom. body we introduce $s(x,t)$ entropy density per unit mass

$$S[\Omega_t] = \int_{\Omega_t} \rho s \, dV_x$$

Generalization of 2nd law to inhom. sys.

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \rho s \, dV_x \geq \int_{\Omega_t} \frac{\rho r}{\theta} dV_x - \int_{\partial\Omega} \frac{q \cdot n}{\theta} \, dA_x}$$

Clausius-Duhem inequality

$\Rightarrow$ places restrictions on constitutive relations.

Note: Precise form of 2nd law in continuum mechanics is not settled yet!

But all proposed forms lead to same restrictions!