Lecture 17: Local Eulerian balance laws

Logistics: - PS7 due Thursday

Last time: - Discrete balance laws

- Balancelaws in integral form $\Sigma \to \int$

  $\Rightarrow$ loose info about velocity fluctuations

  $\Rightarrow$ new continuum variables (T, heat)

- Continuum thermo

  - Net rate of heating: $Q = Q_b + Q_s$
  - Net rate of working: $W = P - \frac{d}{dt} K$
  - First law: $\frac{dU}{dt} = Q + W$
  - Second law: $\frac{d}{dt} S \geq \frac{Q}{\Theta} \leftarrow$ Temp

Today: - local Eulerian balance laws

- mass
- lin / ang. momentum
- energy $\rightarrow$ Networking

I Conservation of mass

Integral form $\frac{d}{dt} M[\Omega_t] = 0$ $\quad M[\Omega_0] = M[\Omega_t]$

Use transform. of volume integrals

$$M[\Omega_t] = \int_{\Omega_t} \rho(x,t)\, dV_x = \int_{\Omega_0} \rho_m(\underline{X},t)\, J(\underline{X},t)\, dV_X$$

where

$$J(X,t) = \det(\underline{F}(\underline{X},t)) \qquad \rho_m(\underline{X},t) = \rho(\varphi(\underline{X},t),t)$$

At $t=0$ $\quad \underline{x} = \underline{X}$ $\quad \Omega_t = \Omega_0$ $\quad J=1$

$$M[\Omega] = \int_{\Omega_0} \rho(x,0)\, dV_x = \int_{\Omega_0} \rho_m(\underline{X},0)\, dV_X = \int \rho_0(\underline{X})\, dV_X$$

$$\rho_0(\underline{X}) = \rho_m(\underline{X},0)$$

Conservation of mass

$$\int_{\Omega_0} \rho_m(X,t)\, J(\underline{X},t) - \rho_0(\underline{X})\, dV_X = 0$$

from arbitraryness of Ω_0

$$\boxed{\rho_m(\underline{X},t)\, J(\underline{X},t) = \rho_0(\underline{X})}$$

Lagrangian statement of mass cons.

Note that the density in ref. conf.

$$\rho_0(X) \neq \rho_m(\underline{X},t)$$

because the volume also changes $J = \frac{dV_x}{dV_X}$

To convert to Eulerian form take $\frac{\partial}{\partial t}$

$$\frac{\partial}{\partial t}\left(\rho_m(\underline{X},t)\, J(\underline{X},t)\right) = \frac{\partial}{\partial t}\rho_0(X) = 0$$

$$\dot{\rho}_m(\underline{X},t)\, J(\underline{X},t) + \rho_m(\underline{X},t)\, \dot{J}(\underline{X},t) = 0$$

$$\dot{J} = J(\nabla_x \cdot \underline{v})_m$$

$$\left[\dot{\rho}_m(X,t) + \rho_m(\nabla_x \cdot \underline{v})_m\right] J(\underline{X},t) = 0$$

divide by J and go to spatial repres.

$$\Rightarrow \quad \boxed{\dot{\rho} + \rho\, \nabla_x \cdot \underline{v} = 0} \quad \text{local Eulerian form}$$

expand material derivative

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla_x \rho + \rho\, \nabla_x \cdot \underline{v} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \underline{v}) = 0}$$

conservative

local Eulerian form

Conservative form:

conserved quantity is in local time deriv.

all fluxes are in divergence

$\Rightarrow$ only an advective flux

Time derivatives of integrals relative to mass / density

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\,\rho(\underline{x},t)\,dV_x = \int_{\Omega_t} \dot{\phi}(\underline{x},t)\,\rho(x,t)\,dV_x}$$

where ϕ is any scalar, vector or tensor field

$$\int_{\Omega_t} \phi\,\rho\,dV_x = \int_{\Omega_0} \phi_m(\underline{X},t)\,\underbrace{\rho_m(\underline{X},t)\,J(X,t)}_{\rho_0(\underline{X})}\,dV_X$$

$$\int_{\Omega_t} \phi\rho\,dV_x = \int_{\Omega_0} \phi_m \rho_0\,dV_X$$

Take the derivative

$$\frac{d}{dt}\int_{\Omega_t} \phi\,\rho\,dV_x = \int_{\Omega_0} \frac{d}{dt}\left(\phi_m(\underline{X},t)\,\rho_0(\underline{X})\right)dV_X$$

$$= \int_{\Omega_0} \dot{\phi}_m \rho_0 \, dV_X$$

$$= \int_{\Omega_0} \dot{\phi}_m \rho_m J \, dV_X$$

$$= \int_{\Omega_t} \dot{\phi}(\underline{x},t) \, \rho(\underline{x},t) \, dV_x$$

$\Rightarrow$ useful for derivation of balance laws

Balance of linear momentum

Integral balance law

$$\frac{d}{dt} \int_{\Omega} \rho \underline{v} \, dV_x = \int_{\partial\Omega} \underline{t} \, dA_x + \int_{\Omega} \rho \underline{b} \, dV_x$$

Cauchy stress: $\underline{t} = \underline{\underline{\sigma}} \, \underline{n}$

$$\frac{d}{dt} \int \rho \underline{v} \, dV_x = \int_{\partial\Omega} \underline{\underline{\sigma}} \, \underline{n} \, dA_x + \int_{\Omega} \rho \underline{b} \, dV_x$$

tensor divergence theorem (Lecture 5)

$$\frac{d}{dt} \int_{\Omega_t} \rho \underline{v} \, dV_x = \int_{\Omega_t} \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} \, dV_x$$

use derivative relative to mass

$$\int_{\Omega_t} \rho \dot{\underline{v}} - \nabla \cdot \underline{\underline{\sigma}} - \rho \underline{b} \, dV_x = 0$$

because Ω_t is arbitrary $\rightarrow$ integrand is zero

$$\boxed{\rho \dot{\underline{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}}$$

local Eulerian form

(Cauchy's first equ of motion)

rewrite in conservative form

$(\underline{v} \cdot \nabla) \underline{v}$

$$\rho \dot{\underline{v}} = \rho \left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v}) \underline{v} \right) = \rho \frac{\partial \underline{v}}{\partial t} + \rho (\nabla_x \underline{v}) \underline{v}$$

$$= \frac{\partial}{\partial t}(\rho \underline{v}) - \underline{v} \frac{\partial \rho}{\partial t} + \rho (\nabla_x \underline{v}) \underline{v}$$

mass balance $\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \underline{v})$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v}) \underline{v} + \rho (\nabla_x \underline{v}) \underline{v}$$

on PS3 $\rightarrow$ $\nabla \cdot (\underline{a} \otimes \underline{b}) = (\nabla \underline{a}) \underline{b} + \underline{a} \, \nabla \cdot \underline{b}$

$a = \underline{v}$ $\quad b = \rho \underline{v}$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v})$$

Conservative Eulerian local balance

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}}$$

conserved quantity: $\rho \underline{v}$ = lin. mom.

advective mom. flux: $\rho \underline{v} \otimes \underline{v}$

diffusive mom. flux: $-\underline{\underline{\sigma}}$

Balance of angular momentum

Integral balance law

$$\frac{d}{dt}\int_{\Omega_t} \underline{x} \times \rho \underline{v} \, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{t} \, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b} \, dA_x$$

lhs:

$$\frac{d}{dt}\int_{\Omega_t} \underbrace{\cancel{\underline{x} \times \rho \underline{v}}}_{}^{\rho(\underline{x}\times\underline{v})} \, dV_x = \int_{\Omega_t} \rho \frac{d}{dt}(\underline{x} \times \underline{v}) \, dV_x$$

$$= \int_{\Omega_t} \rho \left(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}} \right) dV_x$$

$$\dot{\underline{x}} = \underline{v} \rightarrow \underline{v} \times \underline{v} = \underline{0}$$

$$= \int_{\Omega_t} \rho \left(\underline{x} \times \dot{\underline{v}} \right) dV_x$$

rhs $\rightarrow$ subs $\underline{\underline{\sigma}}\,\underline{n} = \underline{t}$

$$\int_{\Omega_t} \rho (\underline{x} \times \dot{\underline{v}}) dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n} \, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b} \, dV_x$$

$$\int_{\Omega_t} \underline{x} \times (\rho \dot{\underline{v}} - \rho \underline{b})\, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n}\, dA_x$$

subst. lin. mom. balance: $\rho \dot{\underline{v}} - \rho \underline{b} = \nabla \cdot \underline{\underline{\sigma}}$

$$\boxed{\int_{\Omega_t} \underline{x} \times \nabla \cdot \underline{\underline{\sigma}}\, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n}\, dA_x}$$

$\Rightarrow$ this is identical to static case (Lecture 7)

$\Rightarrow$ $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ extends to the transient case.

Balance of energy & entropy in Eulerian f.

To use first law of Thermo. need expression for rate of net working.

Power: $P = \underline{f} \cdot \underline{v}$

Newtons 2^{nd}: $\underline{f} = m\underline{a} \rightarrow \underline{f} = \frac{d}{dt}(m\underline{v}) = m\dot{\underline{v}}$

$\Rightarrow P = \underline{f} \cdot \underline{v} = m\dot{\underline{v}} \cdot \underline{v}$

Start with dot product of $\underline{v}$ and $\rho\dot{\underline{v}}$

$$\rho\dot{\underline{v}}\cdot\underline{v} = \rho\underline{v}\cdot\dot{\underline{v}} = (\nabla_x\cdot\underline{\underline{\sigma}})\cdot\underline{v} + \rho\underline{b}\cdot\underline{v}$$

integrate over Ω_t

$$\int_{\Omega_t}\rho\dot{\underline{v}}\cdot\underline{v}\,dV_x = \int_{\Omega_t}(\nabla_x\cdot\underline{\underline{\sigma}})\cdot\underline{v} + \rho\underline{b}\cdot\underline{v}\,dV_x$$

from Lecture 4: $\nabla\cdot(\underline{\underline{A}}^T\underline{b}) = (\nabla\cdot\underline{\underline{A}})\cdot\underline{b} + \underline{\underline{A}}:\nabla\underline{b}$

$$\int_{\Omega_t}\rho\dot{\underline{v}}\cdot\underline{v}\,dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}}:\nabla_x\underline{v} + \nabla\cdot(\underline{\underline{\sigma}}^T\underline{v}) + \rho\underline{b}\cdot\underline{v}\,dV_x$$

use $\underline{\underline{S}}:\underline{\underline{D}} = \underline{\underline{S}}:\mathrm{sym}(\underline{\underline{D}})$ if $\underline{\underline{S}} = \underline{\underline{S}}^T$

here $\underline{\underline{d}} = \mathrm{sym}(\nabla_x\underline{v}) = \frac{1}{2}(\nabla_x\underline{v} + \nabla_x\underline{v}^T)$

$$\int_{\Omega_t}\rho\dot{\underline{v}}\cdot\underline{v}\,dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}}:\underline{\underline{d}} + \rho\underline{b}\cdot\underline{v}\,dV_x + \int_{\partial\Omega}\underline{\underline{\sigma}}\underline{v}\cdot\hat{\underline{n}}\,dA$$

From def. transpose: $\underline{\underline{\sigma}}\underline{v}\cdot\underline{n} = \underline{v}\cdot\underline{\underline{\sigma}}^T\underline{n} = \underline{v}\cdot\underline{t}$

$$\underbrace{\int_{\Omega_t}\rho\dot{\underline{v}}\cdot\underline{v}\,dV_x}_{\frac{d}{dt}K} = \int_{\Omega_t}-\underline{\underline{\sigma}}:\underline{\underline{d}}\,dV_x + \underbrace{\int_{\Omega_t}\rho\underline{b}\cdot\underline{v}\,dV + \int_{\partial\Omega}\underline{v}\cdot\underline{t}\,dA_x}_{P[\Omega_t]}$$

Identify kinetic energy

$$\frac{d}{dt} K[\Omega_t] = \frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \rho \, \underline{v} \cdot \underline{v} \, dV_x = \frac{1}{2} \int \rho \frac{d}{dt} (\underline{v} \cdot \underline{v}) \, dV_x$$

$$\frac{d}{dt} (v_i v_i) = v_i \dot{v}_i + \dot{v}_i v_i = 2 v_i \dot{v}_i$$
$$= 2 \underline{v} \cdot \dot{\underline{v}}$$

$$\frac{d}{dt} K[\Omega_t] = \int_{\Omega_t} \rho \, \underline{v} \cdot \dot{\underline{v}} \, dV_x$$

so we have

$$\frac{d}{dt} K[\Omega_t] + \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x = P[\Omega_t]$$

rate of net working: $W[\Omega_t] = P[\Omega_t] - \frac{d}{dt} K[\Omega_t]$

$$\Rightarrow \quad \boxed{W[\Omega_t] = \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x} \qquad \underline{\underline{d}} = \dot{\underline{\underline{\varepsilon}}}$$

the quantity $\underline{\underline{\sigma}} : \underline{\underline{d}}$ is called the stress power of a motion. The rate of

work done by the internal forces (stresses)
in a continuum body.