Lecture 18: Continuum Thermo & Lagrangian Bal.

Logistics: - PS 7 is due

- probably no new PS this week

Last time: Eulerian balance laws

$$\frac{d}{dt}\int_{\Omega_t}\rho\phi\, dV_x = \int_{\Omega_t}\rho\dot{\phi}\, dV_x$$

mass: $\dot{\rho} + \rho\nabla_x\cdot\underline{v} = 0$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$$

lin. mom: $\rho\dot{\underline{v}} = \nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b}$

$$\frac{\partial}{\partial t}(\rho\underline{v}) + \nabla\cdot(\rho\,\underline{v}\otimes\underline{v} - \underline{\underline{\sigma}}) = \rho\underline{b}$$

ang. mom: $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$

rate of net working:

$$\boxed{\begin{aligned}\frac{d}{dt}K[\Omega_t] &= \int_{\Omega_t}\rho\,\underline{v}\cdot\dot{\underline{v}}\, dV \\ P[\Omega_t] &= \int_{\Omega_t}\rho\underline{b}\cdot\underline{v}\, dV + \int_{\partial\Omega_t}\underline{t}\cdot\underline{v}\, dA_x \\ W[\Omega_t] &= \int_{\Omega_t}\underline{\underline{\sigma}}:\underline{\underline{d}}\, dV_x\end{aligned}}$$

Today: Eulerian Energy & Entropy balance

Lagrangian balance laws

Local Eulerian form of First law of Thermo.

Integral form

$$\frac{d}{dt}\,\mathcal{U}[\Omega_t] = Q[\Omega_t] + W[\Omega_t]$$

where

$$\mathcal{U}[\Omega_t] = \int_{\Omega} \rho \overset{u}{\phi}\, dV_x$$

$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} q \cdot \underline{n} \, dA_x$$

$$W[\Omega_t] = \int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x$$

Substituting

$$\frac{d}{dt}\int_{\Omega_t} \rho\phi \, dV_x = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x - \int_{\partial\Omega_t} q \cdot n \, dA_x + \int_{\Omega_t} \rho r \, dV_x$$

using deriv. rel. to mass & div. theorem

$$\int_{\Omega_t} \rho\dot{\phi} - \underline{\underline{\sigma}} : \underline{\underline{d}} + \nabla_x \cdot q - \rho r \, dV_x = 0$$

by arbitrary ness of Ω_t

$$\boxed{\rho\dot{\phi} = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla_x \cdot q + \rho r}$$

local eulerian form of energy bal.

for cons. form expand $\dot{\phi} = \frac{\partial\phi}{\partial t} + \underline{v} \cdot \nabla\phi$

$$\rho\dot{\phi} = \rho\frac{\partial\phi}{\partial t} + \rho\,\underline{v}\cdot\nabla_x\phi = \frac{\partial}{\partial t}(\rho\phi) - \phi\frac{\partial\rho}{\partial t} + \rho\,\underline{v}\cdot\nabla\phi$$

mass balance: $\frac{\partial\rho}{\partial t} = -\nabla_x\cdot(\rho\underline{v})$

$$\rho\dot{\phi} = \frac{\partial}{\partial t}(\rho\phi) + \phi\nabla_x\cdot(\rho\underline{v}) + \rho\,\underline{v}\cdot\nabla\phi$$

$$= \frac{\partial}{\partial t}(\rho\phi) + \nabla_x\cdot(\rho\phi\underline{v})$$

$\Rightarrow$ subst. conservative form

$$\boxed{\frac{\partial}{\partial t}(\rho\phi) + \nabla_x\cdot[\rho\phi\underline{v} + q] = \underline{\underline{\sigma}}:\underline{\underline{d}} + \rho r}$$

Local Eulerian form of Second Law

Integral form of Clausius-Duhem form of 2$^{\text{nd}}$ law

$$\frac{d}{dt}\int_{\Omega_t}\rho s\,dV_x \geq \int_{\Omega_t}\frac{\rho r}{\theta}\,dV_x - \int_{\partial\Omega_t}\frac{q\cdot n}{\theta}\,dA_x$$

using deriv. with respect to mass + div thm

$$\int_{\Omega_t}\rho\dot{s}\,dV_x \geq \int_{\Omega}\frac{\rho r}{\theta} - \nabla_x\cdot\frac{q}{\theta}\,dV_x$$

$\rightarrow$ localizing

$$\boxed{\rho\dot{s} \geq \frac{\rho r}{\theta} - \nabla_x\cdot\left(\frac{q}{\theta}\right)}$$

Clausius-Duhem in local Eulerian form

$$\theta \rho \dot{s} \geq \rho r - \theta \left(\frac{1}{\theta} \nabla_x \cdot q + q \cdot \nabla_x \left(\frac{1}{\theta} \right) \right)$$

$$\geq \rho r - \nabla_x \cdot q + \frac{1}{\theta} \nabla_x \theta$$

introduce internal dissipation density

$$\delta = \theta \rho \dot{s} - (\rho r - \nabla_x \cdot q)$$

difference between local entropy increase and local rate of heating $\quad dS \geq \frac{dQ}{\theta}$

$$\boxed{\delta - \frac{1}{\theta} q \cdot \nabla_x \theta \geq 0}$$

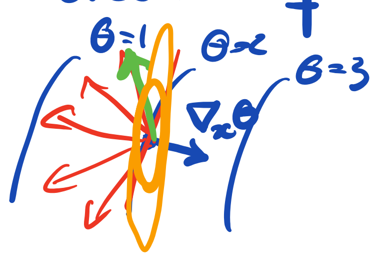
I, Any point where $\nabla_x \theta = 0 \Rightarrow \delta \geq 0$

$\Rightarrow$ bodies with uniform θ have non. neg. dissip.

II If $\delta = 0$, i.e. reversible process

$\Rightarrow \quad q \cdot \nabla_x \theta \leq 0$

angle between q and $\nabla_x \theta > 90$



q = energy density

$\Rightarrow$ heat flows down temp. gradient

Fourier's law: $\quad \underline{q} = -\underline{\underline{\kappa}} \nabla \theta$

To study the consequences of Clausius-Duhem for constitutive laws we introduce Helmholtz free energy density:

$$\psi(\underline{x},t) = \phi(\underline{x},t) - \theta(\underline{x},t)\, s(\underline{x},t)$$

$$H \quad = U - TS$$

Reformulate CD ~~for~~ in terms of ψ

Material derivative

$$\frac{d}{dt}(\theta s) = \frac{\partial}{\partial t}(\theta s) + \nabla_x(\theta s)\cdot \underline{v}$$

$$= \theta \frac{\partial s}{\partial t} + s\frac{\partial \theta}{\partial t} + \theta\, \nabla_x s\cdot \underline{v} + s \nabla_x \theta \cdot \underline{v}$$

$$= \theta\left(\frac{\partial s}{\partial t} + \nabla_x s \cdot \underline{v}\right) + s\left(\frac{\partial \theta}{\partial t} + \nabla_x \theta \cdot \underline{v}\right)$$

$$= \theta \dot{s} + s\dot{\theta}$$

from def. $\quad \psi = \phi - \theta s$

$$\dot\psi = \dot\phi - \theta\dot s - s\dot\theta \Rightarrow \dot\phi = \dot\psi + \theta\dot s + s\dot\theta$$

subst into energy cons.

$$\rho\dot\phi = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla_x \cdot q + \rho r$$

$$\rho\dot\psi + \rho\theta\dot s + \rho s\dot\theta =$$

$$\rho\theta\dot s = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla_x \cdot q + \rho r - \rho\dot\psi - \rho s\dot\theta$$

substitute into 2nd law

$$\rho\theta\dot s \geq \rho r - \nabla_x \cdot q + \frac{1}{\theta} q \cdot \nabla_x\theta$$

$$\underline{\underline{\sigma}} : \underline{\underline{d}} - \cancel{\nabla_x \cdot q} + \cancel{\rho r} - \rho\dot\psi - \rho s\dot\theta \geq \cancel{\rho r} - \cancel{\nabla_x \cdot q} + \frac{1}{\theta} q \cdot \nabla_x\theta$$

solve $\rho\dot\psi$

$$\boxed{\rho\dot\psi \leq \underline{\underline{\sigma}} : \underline{\underline{d}} - \rho s\dot\theta - \frac{1}{\theta} q \cdot \nabla_x\theta}$$

reduced Clausius-Duhem inequality

- independent of q and r

In a body with const. θ $\quad \dot\theta = 0 \quad \nabla_x\theta = 0$

$$\boxed{\rho\dot\psi \leq \underline{\underline{\sigma}} : \underline{\underline{d}}}$$

for a reversible process $\Rightarrow$ $\rho\dot\psi = \underline{\underline{\sigma}} : \underline{\underline{d}}$

$\Rightarrow$ rate of change of Helmholtz free energy

is equal to stress power in rev. process
$\Rightarrow$ Helmholtz free energy is the part of the internal energy available for performing work at const. θ.

Summary Eulerian Balance Laws

Governing equations:

mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$ 1

lin. mom: $\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}$ 3

ang. mom: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ 3

energy: $\frac{\partial}{\partial t}(\rho \phi) + \nabla \cdot (\rho \phi \underline{v} + \underline{q}) = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r$ 1

kinematic: $\underline{v}_m = \frac{\partial \underline{\varphi}}{\partial t}$ + 3

11

Unknown fields

$\underline{\varphi}$	$\underline{v}$	ρ	$\underline{\underline{\sigma}}$	θ	$\underline{q}$	ϕ	
3	3	1	9	1	3	1	= 21

We have 21 unknown fields but only 11 equations.

$\Rightarrow$ under constrained

$\Rightarrow$ Need additional constitutive eqns

Remarks:

1) Eulerian formulation of balance laws is independent of φ!
The motion is only needed to determine shape of domain. If B_t is known we 8 eqns and 18 unknowns
Need constitutive eqns that relate

secondary $\qquad\qquad\qquad\qquad$ primary unknowns

$\underline{\underline{\sigma}}$, $\underline{q}$ $\underline{\Phi}$ to ρ $\underline{v}$ and θ

2) If thermal effects are neglected:
q ϕ & θ dissapear reducing the unknowns to 12. The number of eqns reduces

to 7. Need constitutive eqns that relate $\underline{\sigma}$ to ρ and $\underline{v}$.