Lecture 19: Lagrangian balance laws

Logistics: —

Last time: - Continuum thermo

- 1^{st} law: $\rho\dot{\phi} = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla_x \cdot q + \rho r$

  $$\frac{\partial}{\partial t}(\rho\phi) + \nabla \cdot (\rho\phi\underline{v} + q) = \underline{\underline{\sigma}} : \underline{\underline{d}} - \rho r$$

- 2^{nd} law: $\rho\dot{s} \geq \frac{\rho r}{\theta} - \nabla_x \cdot \left(\frac{q}{\theta}\right)$ C-D

  $\Rightarrow$ show dir. of heat conduction

- Helmholtz free energy: $\psi = \phi - \theta s$

  $\Rightarrow \rho\dot{\psi} \leq \underline{\underline{\sigma}} : \underline{\underline{d}}$ if $\theta = \text{const.}$

  stress power is rate of change of free energy.

  $\Rightarrow \psi$ is part of ϕ available to do work

- Eulerian form: 21 unknowns & 11 eqns !

Today: - Lagrangian balance laws

$\underline{x} \longrightarrow \underline{X}$

E $\quad$ L

Balance of mass

already showed this

$$\boxed{\rho_m(X) \det \underline{\underline{F}}(\underline{X},t) = \rho_0(X)}$$

if φ is known $\rightarrow F = \nabla\varphi \rightarrow \rho_m$ is know

mass density is a known quantity

Balance of lin. momentum

Integral balance law

$$\frac{d}{dt}\underline{L}[\Omega_t] = \underline{r}[\Omega_t]$$

where $\underline{L}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, \underline{v}(\underline{x},t)\, dt$

$$\underline{r}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, \underline{b}(\underline{x},t)\, dV_x + \int_{\partial\Omega_t} \underline{\underline{\sigma}}\, \underline{n}\, dA_x$$

change variables $\underline{x} \rightarrow \underline{X}$

$$\underline{L}[\Omega_t] = \int_{\Omega_0} \rho_m(\underline{X},t)\, \underbrace{\underline{v}_m(\underline{X},t)}_{}\, \underbrace{\det F(\underline{X},t)}_{J}\, dV_X$$

$$\rho_0(\underline{X}) = \rho_m J \qquad \underline{V}(\underline{X},t) = \dot{\underline{\varphi}}$$

$$L[\Omega_t] = \int_{\Omega_0} \rho_0(\underline{X})\, \dot{\varphi}(\underline{X},t)\, dV_X$$

To change surface integral Nansen's form.

$$\underline{n}\, dA_x = J\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X$$

$$\underline{\Gamma}[\Omega_t] = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\, \underline{n}\, dA_x + \int_{\Omega_t} \rho\, \underline{b}\, dV_x$$

$$= \int_{\partial\Omega_0} \underline{\underline{\sigma}}_m \det(\underline{\underline{F}})\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X + \int_{\Omega_0} \rho_m \underline{b}_m J\, dV_X$$
$$+ \int_{\Omega_0} \rho_0 \underline{b}_m\, dV_X$$

Introduce first Piola-Kirchhoff stress tensor

$$\boxed{\underline{\underline{P}}(\underline{X},t) = \det \underline{\underline{F}}\; \underline{\underline{\sigma}}_m\, \underline{\underline{F}}^{-T}}$$

using $\underline{\underline{P}}$ we have

$$\underline{\Gamma}[\Omega_t] = \int_{\partial\Omega_0} \underline{\underline{P}}\,\underline{N}\, dA_X + \int_{\Omega_0} \rho_0 \underline{b}_m\, dV_X$$

hence we have

$$\frac{d}{dt}\int_{\Omega_0} \rho_0 \dot{\underline{\varphi}}\, dV_X = \int_{\partial\Omega_0} \underline{\underline{P}}\,\underline{N}\, dA_X + \int_{\Omega_0} \rho_0 \underline{b}_m\, dV_X$$

$$\int_{\Omega} \rho_0 \ddot{\underline{\varphi}}\, dV_X = \int_{\Omega_0} \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m\, dV_X$$

hence

$$\boxed{\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m}$$ local Lagrangian lin. moment. balance

Note: $\underline{\underline{P}}$ is the natural stress tensor in the material description. It relates the traction to normal of surface

spatial:

$$\boxed{\begin{aligned} \underline{t}(\underline{x},t) &= \underline{\underline{\sigma}}(\underline{x},t)\,\underline{n}(\underline{x},t) \\ \underline{T}(\underline{X},t) &= \underline{\underline{P}}(\underline{X},t)\,\underline{N}(\underline{X},t) \end{aligned}}$$

Here $\underline{T}$ is the (nominal) Piola-Kirchhoff traction vector and $\underline{t}$ (true) Cauchy traction vector.

$$\underline{df} = \underline{t}\, dA_x = \underline{T}\, dA_X$$

$\Rightarrow$ $\underline{t} \parallel \underline{T}$ same direction different magn.

Balance of angular momentum

from Eulerian form: $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T \Rightarrow \underline{\underline{\sigma}}_m = \underline{\underline{\sigma}}_m^T$

def. $\underline{\underline{P}} = J\, \underline{\underline{\sigma}}_m \underline{\underline{F}}^{-T}$

$$\underline{\underline{\sigma}}_m = \frac{1}{J} \underline{\underline{P}}\, \underline{\underline{F}}^T = \underline{\underline{\sigma}}_m^T = \left(\frac{1}{J} \underline{\underline{P}}\, \underline{\underline{F}}^T\right)^T = \frac{1}{J} \underline{\underline{F}}\, \underline{\underline{P}}^T$$

$\Rightarrow$ $\boxed{\underline{\underline{P}}\, \underline{\underline{F}}^T = \underline{\underline{F}}\, \underline{\underline{P}}^T}$ constraint from angular mom. balance

$\Rightarrow$ $\underline{\underline{P}} \neq \underline{\underline{P}}^T$ $\Rightarrow$ $\underline{\underline{P}}$ has 9 indep. components.

ang. mom. balance motivates second Piola-Kirchhoff stress tensor

$\boxed{\underline{\underline{\Sigma}} = \underline{\underline{P}}\, \underline{\underline{F}}^T}$ so that $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^T$

Char. of Net working

$$\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt}\mathcal{K}[\Omega_t]$$

$$\mathcal{K}[\Omega_t] = \int_{\Omega_t} \frac{1}{2}\rho |\underline{v}|^2 \, dV_x$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega_t} \rho b \cdot \underline{v} \, dV_x + \int_{\partial\Omega} \underline{\underline{\sigma}}\,\underline{v} \cdot \underline{n} \, dA_x$$

used transpose: $\underline{v} \cdot \underline{\underline{\sigma}}\,\underline{n} = \underline{\underline{\sigma}}^T \underline{v} \cdot \underline{n} = \underline{\underline{\sigma}}\,\underline{v} \cdot \underline{n}$

Changing variables:

$$\mathcal{K}[\Omega_t] = \int_{\Omega_0} \frac{1}{2}\rho_0 |\dot{\varphi}|^2 \, dV_X$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega_0} \rho_0 \underline{b}_m \cdot \dot{\varphi} \, dV_X + \int_{\partial\Omega_0} J \, \underline{\underline{\sigma}}_m \underline{v}_m \cdot \underline{\underline{F}}^{-T} \underline{N} \, dA_X$$

$$+ \int J \, \underline{v}_m \cdot \underline{\underline{\sigma}}_m F^{-T} \underline{N} \, dA_X$$

$$+ \int \dot{\varphi} \cdot \underbrace{J \, \underline{\underline{\sigma}}_m F^{-T}}_{\underline{\underline{P}}} \underline{N} \, dA_X$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega_0} \rho_0 \underline{b}_m \cdot \dot{\varphi} \, dV_X + \int_{\partial\Omega_0} \dot{\varphi} \cdot \underline{\underline{P}}\, \underline{N} \, dA_X$$

$$= \int_{\Omega} \rho_0 \underline{b}_m \cdot \dot{\varphi} \, dV_X + \int_{\partial\Omega_0} \underline{\underline{P}}^T \dot{\varphi} \cdot \underline{N} \, dA_X$$

use identity: $\nabla_X \cdot (\underline{\underline{P}}^T \dot{\varphi}) = (\nabla_X \cdot \underline{\underline{P}}) \cdot \dot{\varphi} + \underline{\underline{P}} : \underbrace{\nabla \dot{\varphi}}_{\dot{\underline{\underline{F}}}}$

Lagrangian expression for power.

$$\boxed{\mathcal{P}[\Omega_t] = \int_{\Omega_0} [\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m] \cdot \dot{\varphi} + \underline{\underline{P}} : \dot{\underline{\underline{F}}} \; dV_X}$$

The rate of change of kinetic energy

$$\frac{d}{dt} \mathcal{K}[\Omega_t] = \int_{\Omega} \frac{1}{2} \rho_0 \frac{d}{dt} |\dot{\varphi}|^2 \, dV_X = \int \dot{\varphi} \cdot (\rho_0 \ddot{\varphi}) \, dV_X$$

use lin. mom. balance $\rho_0 \ddot{\varphi} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m$

$$\frac{d}{dt} \mathcal{K}[\Omega_t] = \int_{\Omega} [\nabla_X \cdot \underline{\underline{P}} + \rho_0 b_m] \cdot \dot{\varphi} \, dV_X$$

Hence net working

$$\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt} \mathcal{K}[\Omega_0] = \int_{\Omega_0} \underline{\underline{P}} : \dot{F} \; dV_X$$

Lagrangian def net working

$$\boxed{\mathcal{W}[\Omega_t] = \int_{\Omega_0} \underline{\underline{P}} : \dot{\underline{\underline{F}}} \; dV_X}$$

$\underline{\underline{P}} : \dot{\underline{\underline{F}}}$ is the Lagrangian stress power

$\underline{\underline{\sigma}} : \underline{\underline{d}}$ is the Eulerian stress power

$\underline{\underline{P}} : \dot{\underline{\underline{F}}}$ is power per unit vol. B_0

$\underline{\underline{\sigma}} : \underline{\underline{d}}$ is power per unit vol. B_t

Lagrangian Energy balance

Integral form of 1st Law of thermo

$$\frac{d}{dt} \mathcal{U}[\Omega_t] = Q[\Omega_t] + \mathcal{W}[\Omega_t]$$

where

$$\mathcal{U}[\Omega_t] = \int_{\Omega_t} \rho u \, dV_x \qquad \phi \equiv u$$

$$\int_{\Omega_0} \rho_0 u_m \, dV_X \qquad U = u_m$$

$$\int_{\Omega_0} \rho_0 U \, dV_X$$

Rate of net heating

$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} q \cdot n \, dA_x$$

Nanson's formula: $\underline{n}\, dA_x = J\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X$

$$Q[\Omega_t] = \int_{\Omega_0} \rho_0 R\, dV_X - \int_{\partial\Omega_0} q_m \cdot J\, \underline{\underline{F}}^{-T} \underline{N}\, dA_X$$

$R = r_m$

use transpose : $\underbrace{J F^{-1} q_m}_{\underline{Q}} \cdot N = \underline{q_m} \cdot J F^{-T} \underline{N}$

$\underline{Q}$ material heat flux

Lag. net heating

$$Q[\Omega_t] = \int_{\Omega_0} \rho_0 R\, dV_X - \int_{\partial\Omega_0} \underline{Q} \cdot \underline{N}\, dA_X$$

subst. into 1^{st} law + div. thm + localization

$$\boxed{\rho_0 \dot{U} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot \underline{Q} + \rho_0 R}$$ local Lagrangian energy balance

Lagrangian form 2^{nd} law

Integral balance

$$\int_{\Omega_t} \rho\, \dot{s}\, dV_x \geq \int_{\Omega_t} \frac{\rho r}{\Theta}\, dV_x - \int \frac{q \cdot n}{\Theta}\, dA_x$$

change variables

$$\int_{\Omega_0} \rho_0 S \, dV_X \geq \int_{\Omega_0} \frac{\rho_0 R}{\Theta} dV_X - \int_{\partial\Omega_0} \frac{\underline{Q} \cdot \underline{N}}{\Theta} dA_X$$

$S = s_m \qquad \Theta = \theta_m$

using div. theorem + localization.

$$\boxed{\rho_0 \dot{S} \geq \frac{\rho_0 R}{\Theta} - \nabla_X \cdot \frac{\underline{Q}}{\Theta}}$$

Clausius–Duhem in Lag. form

Introduce Helmholtz free energy

$$\Psi(\underline{X}, t) = U(\underline{X}, t) - \Theta(\underline{X}, t) S(\underline{X}, t)$$

similar to Eulerian case

$$\rho_0 \dot{\Psi} \leq \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \rho_0 S \dot{\Theta} - \frac{1}{\Theta} \underline{Q} \cdot \nabla_X \Theta$$

reduced C-D inequality

$$\Rightarrow \quad \rho_0 \dot{\Psi} = \underline{\underline{P}} : \dot{\underline{\underline{F}}}$$

Summary of Lagrangian formulation

Balance laws

Kinematic: $\underline{V} = \dot{\underline{\varphi}}$ 3

lin. mom: $\rho_0 \dot{\underline{V}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 b_m$ 3

ang. mom: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^T \quad \underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$ 3

energy bal: $\rho_0 \dot{U} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot \underline{Q} + \rho_0 R$ + 1

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Lagrangian fields

$\underline{\varphi}$	$\cancel{\underline{V}}$	$\underline{\underline{P}}$	$\cancel{\Theta}$	$\cancel{\underline{Q}}$	$\cancel{U}$	
3	$\cancel{3}$	9	$\cancel{1}$	$\cancel{3}$	$\cancel{1}$	$\hat{=}\ 20 - 3 - 1 - 3 - 1$ (12)

We have 20 unknowns and 10 equations

$\Rightarrow$ Lag. form. has one less unknown & equ compared to Eulerian, because ρ is known

Remarks:

1) In many situations $\underline{V}$ is not needed explicitly

$\Rightarrow$ 17 unknowns & 7 equs

2) Need const. equations that relate

$\underline{\underline{\Sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T$, $\underline{Q}$, $\underline{u}$ to φ and Θ

3) If thermal effects are also neglected

$\Rightarrow$ 12 unknowns and 6 eqns

to close system requires 6 eqns

relating $\underline{\underline{\Sigma}}$ to φ