Lecture 19: Frame indifference/Objectivity

Logistics: - Hope to post PS 8 today
- Hand in missing PS !

Last time: Lagrangian balance laws

mass: $\rho_m J = \rho_0$

lin. mom.: $\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 b_m$ $\qquad \dot{\underline{\varphi}} = \underline{V} = \underline{A}$

ang. mom.: $\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$ $\qquad \underline{\underline{P}}^T \neq \underline{\underline{P}}$

energy: $\rho_0 \dot{U} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot \underline{Q} + \rho_0 R$

entropy: $\rho_0 \dot{S} \geq \dfrac{\rho_0 R}{\Theta} - \nabla_X \cdot \dfrac{\underline{Q}}{\Theta}$

1st Piola-Kirchhoff stress: $\underline{\underline{P}} = J \underline{\underline{\sigma}}_m \underline{\underline{F}}^{-T}$

2nd Piola-Kirchhoff stress: $\underline{\underline{\Sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^{-T} = \underline{\underline{\Sigma}}^T$

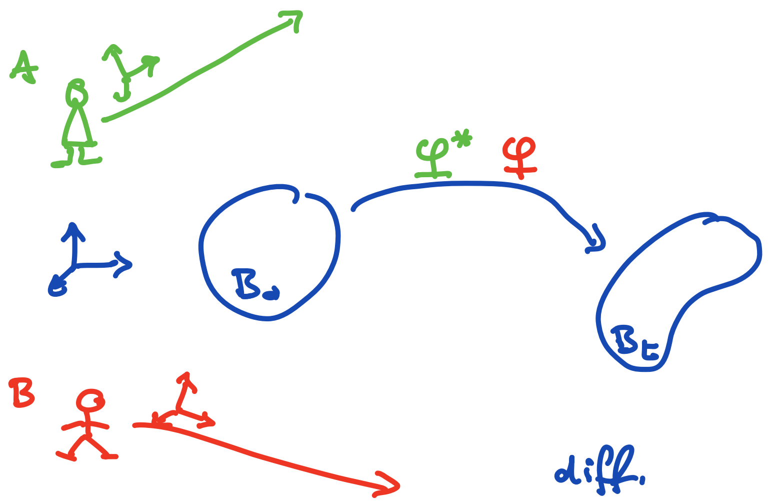
$d\underline{f} = \underline{t}\, dA_x$
$= \underline{T}\, dA_X$

Piola-Kirchhoff traction: $\underline{T} = \underline{\underline{P}}\,\underline{N}$
$\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$

Today: Constitutive theory

Frame indifference

Frame Indifference / Objectivity



Two observers in diff. frames $\{\underline{e}_i\}$ and $\{\underline{e}_i^*\}$ must be related by a rigid motion in a common frame

$$\boxed{\begin{aligned} \underline{x}^* = \varphi^*(\underline{X}, t) &= \underline{\underline{Q}}(t)\,\varphi(\underline{X}, t) + \underline{c}(t) \\ \underline{x}^* &= \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t) \end{aligned}}$$

Euclidean transformation

where $Q(t)$ is a rotation and $c(t)$ is a translation of obs. A relative to obs. B

Note: We assume they are on same clock.

Effect of superposed rigid motion on kinematic fields:

Material fields:

$$\boxed{\begin{array}{lll} \underline{\underline{F}}^* = \underline{\underline{Q}}\,\underline{\underline{F}} & \underline{\underline{R}}^* = \underline{\underline{Q}}\,\underline{\underline{R}} & \\ \underline{\underline{U}}^* = \underline{\underline{U}} & \underline{\underline{V}}^* = \underline{\underline{Q}}\,\underline{\underline{V}}\,\underline{\underline{Q}}^T & \underline{\underline{C}}^* = \underline{\underline{C}} \end{array}}$$

note $\underline{X}^* = \underline{X}$ material frame is independent of the observer. The spatial coordinates in contrast are different: $\underline{x} = \underline{\varphi}(\underline{X}, t)$

$$\underline{x}^* = \underline{\varphi}^*(\underline{X}, t)$$

Spatial fields:

$$\boxed{\begin{array}{l} \underline{\underline{\ell}}^* = \nabla_{\underline{x}^*} \underline{v}^* = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T \\ \underline{\underline{d}}^* = \mathrm{sym}(\underline{\underline{\ell}}^*) = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T \end{array}}$$

$$\dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T = \underline{\underline{\Omega}}$$

where $\underline{\underline{\Omega}}$ is the spin of $\{\underline{e}_i^*\}$ relative to $\{\underline{e}_i\}$ $\Rightarrow$ skew $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$

$\dot{Q} = \frac{\partial}{\partial t} Q(t)$ because $Q \neq Q(\underline{X})$

Example: $$\varphi = \underline{\underline{F}}(\underline{X},t)\,\underline{X}$$

$$\varphi^* = \underline{\underline{Q}}\,\varphi + \underline{c} = \underline{\underline{Q}}\,\underline{\underline{F}}\,\underline{X} + \underline{c}$$

$$\underline{\underline{F}}^* = \nabla \varphi^* = \underline{\underline{Q}}\,\underline{\underline{F}}$$

$$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$$

$$\underline{\underline{C}}^* = \underline{\underline{F}}^{*T} \underline{\underline{F}}^* = (\underline{\underline{Q}}\,\underline{\underline{F}})^T \underline{\underline{Q}}\,\underline{\underline{F}} =$$

$$= \underline{\underline{F}}^T \underbrace{\underline{\underline{Q}}^T \underline{\underline{Q}}}_{\underline{\underline{I}}}\,\underline{\underline{F}} = \underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{C}}$$

From Lecture 15: $$\underline{\underline{l}} = \nabla_x \underline{v}\big|_{\underline{x}=\varphi(\underline{X},t)} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$$

$$\underline{\underline{l}}^* = \nabla_{x^*} \underline{v}^*\big|_{\underline{x}^*=\varphi^*(\underline{X},t)} = \dot{\underline{\underline{F}}}^*\,\underline{\underline{F}}^{*-1}$$

$$\dot{\underline{\underline{F}}}^* = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}$$

$$\underline{\underline{F}}^{*-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\underline{\underline{l}}^* = \dot{\underline{\underline{F}}}^*\,\underline{\underline{F}}^{*-1} = (\dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \underbrace{\dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T}_{\underline{\underline{\Omega}}} + \underline{\underline{Q}}\,\underbrace{\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}}_{l}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{l}}\,\underline{\underline{Q}}^T + \underline{\underline{\Omega}}$$

Axiom of frame indifference

Fields ϕ, $\underline{w}$, $\underline{\underline{S}}$ are called frame indifferent if for all superimposed rigid motions

$\underline{x}^* = \underline{\underline{Q}}\,\underline{x} + c$ we have

$$\boxed{\begin{aligned} \phi^*(\underline{x}^*, t) &= \phi(\underline{x}, t) \\ \underline{w}^*(\underline{x}^*, t) &= \underline{\underline{Q}}\,\underline{w}(\underline{x}, t) \\ \underline{\underline{S}}^*(\underline{x}^*, t) &= \underline{\underline{Q}}\,\underline{\underline{S}}(\underline{x}, t)\,\underline{\underline{Q}}^T \end{aligned}}$$

Some field, such as density or temperature, are inherent to body and independent of observer. Not all fields are frame indifferent. Two observers in relative motion to each other will disagree on velocity & acc.

Tensors that are frame invariant are $\underline{\underline{d}}$, $\underline{v}$, called objective.

Example:

Newtonian Stress: $\sigma \neq 2\mu \nabla_x \underline{v}$

$$\sigma = 2\mu \, \mathrm{sym}(\nabla_x \underline{v})$$

$$= \mu(\nabla_x \underline{v} + \nabla_x v^T)$$

$\Rightarrow$ why constitutive laws are based on symmetric parts of strain or rate of strain tensors.

<u>Galilean transformations</u>

Consider velocity under change of observer.

$$\underline{x} = \varphi(X,t)$$

$$\underline{v}(\underline{x},t) = V(\underline{X},t) = \dot{\varphi}(X,t)$$

$$\underline{v}^*(\underline{x}^*,t) = V^*(\underline{X},t) = \dot{\varphi}^*(X,t)$$

$$= \frac{d}{dt}(\underline{Q}\varphi + \underline{c}) = \dot{\underline{Q}}\varphi + \underline{Q}\dot{\varphi} + \dot{\underline{c}} \qquad \dot{\varphi} = \underline{v}$$

$$\underline{v}^* = Q\underline{v} + \dot{Q}\underline{x} + \dot{\underline{c}}$$

$\underline{v}$ is only objective $\dot{Q} = \dot{c} = 0$

substituting: $\underline{x}^* = \underline{\underline{Q}}\,\underline{x} + \underline{c}$

$$\frac{d}{dt}\left(\underline{x} = \underline{\underline{Q}}^T(\underline{x}^* - \underline{c})\right)$$

$$\underline{v}^* = \underline{\underline{Q}}\,\underline{v} + \underbrace{\dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T}_{\underline{\underline{\Omega}}}(\underline{x}^* - \underline{c}) + \dot{\underline{c}}$$

$$\boxed{\underline{v}^* = \underline{\underline{Q}}\,\underline{v} + \dot{\underline{c}} + \underbrace{\underline{\underline{\Omega}}(\underline{x}^* - \underline{c})}_{\text{Euler acceleration}}}$$

to get acceleration.

$$\underline{a}^* = \dot{\underline{v}}^* = \underline{\underline{Q}}\,\underset{\underline{a}}{\dot{\underline{v}}} + \ddot{\underline{c}} + \dot{\underline{\underline{Q}}}\,\underline{v} + \dot{\underline{\underline{\Omega}}}(\underline{x}^* - c) + \underline{\underline{\Omega}}(\dot{\underline{x}}^* - \dot{\underline{c}})$$

subst. $\underline{v} = \dot{\underline{\underline{Q}}}^T(\underline{x}^* - c) + \underline{\underline{Q}}^T(\dot{\underline{x}}^* - \dot{\underline{c}})$

$$\underline{\underline{\Omega}}^2 = -\dot{\underline{\underline{Q}}}\,\dot{\underline{\underline{Q}}}^T$$

$$\boxed{\underline{a}^* = \underline{\underline{Q}}\,\underline{a} + \ddot{\underline{c}} + (\dot{\underline{\underline{\Omega}}} - \underline{\underline{\Omega}}^2)(\underline{x}^* - \underline{c}) + 2\underline{\underline{\Omega}}(\underline{v} - \dot{\underline{c}})}$$

$\Rightarrow$ in general $\underline{a}$ is not frame indifferent

3 fictitious accelerations:

1) Euler acceleration: $\dot{\underline{\underline{\Omega}}}(\underline{x}^* - \underline{c})$

2) Centrifugal acc. : $-\underline{\underline{\Omega}}^2(\underline{x}^* - \underline{c})$

3) Coriolis acc. : $2\underline{\underline{\Omega}}(\underline{v}^* - \dot{\underline{c}})$

Acceleration is only objective if $\ddot{\underline{c}} = \underline{0}$ $\dot{\underline{\underline{Q}}} = \underline{\underline{0}}$

$$\boxed{\underline{x}^* = \underline{\underline{Q}}\,\underline{x} + \underbrace{\underline{c}_0 + v_0 t}_{c(t)}}$$

Galilean transformations.

Two observers moving relative to each other with const. velocity measure the same acceleration.

Frame indifferent functions

Mass density ρ, temperature θ, energy & entropy densities u & s are scalars and frame indifferent in addition Cauchy stress $\underline{\underline{\sigma}}$ and heat flow $\underline{q}$ are also frame indifferent.

Hence constitutive functions.

$$\phi(\underline{x}, t) = \hat{\phi}(\rho(\underline{x}, t), \theta(\underline{x}, t), \underline{\underline{S}}(\underline{x}, t))$$

$$\underline{q}(\underline{x}, t) = \hat{\underline{q}}(\rho, \theta, \underline{\underline{S}})$$

$$\underline{\underline{\sigma}}(\underline{x}, t) = \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{S}})$$

To be frame indifferent requires

$$\phi(\underline{x}, t) = \phi^*(\underline{x}^*, t)$$

$$\hat{\phi}(\rho^*, \theta^*, \underline{\underline{S}}^*) = \hat{\phi}(\rho, \theta, \underline{\underline{S}})$$

since $\rho^* = \rho \quad \theta^* = \theta \quad \underline{\underline{S}}^* = \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T$

$$\Rightarrow \quad \hat{\phi}(\rho, \theta, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\rho, \theta, \underline{\underline{S}})$$

This implies any such function can only depend on invariants of $\underline{\underline{S}}$

For vector function

$$\hat{\underline{q}}(\rho^*, \theta^*, \underline{\underline{S}}^*) = \underline{\underline{Q}}\,\hat{\underline{q}}(\rho, \theta, \underline{\underline{S}})$$

$$\hat{\underline{q}}(\rho, \theta, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{q}}(\rho, \theta, \underline{\underline{S}})$$

For tensor function

$$\hat{\underline{\underline{\sigma}}}(\rho^*, \underline{\theta}^*, \underline{\underline{S}}^*) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{S}})\,\underline{\underline{Q}}^T$$

$$\hat{\underline{\underline{\sigma}}}(\rho, \underline{\theta}, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{S}})\,\underline{\underline{Q}}^T$$

In summary material frame indifference requires that

$$\boxed{\begin{aligned}
\hat{\phi}(p,\theta,\underline{\underline{Q}}\,\underline{\underline{s}}\,\underline{\underline{Q}}^T) &= \hat{\phi}(p,\theta,\underline{\underline{s}}) \\
\hat{\underline{q}}(p,\theta,\underline{\underline{Q}}\,\underline{\underline{s}}\,\underline{\underline{Q}}^T) &= \underline{\underline{Q}}\,\hat{\underline{q}}(p,\theta,\underline{\underline{s}}) \\
\hat{\underline{\underline{s}}}(p,\theta,\underline{\underline{Q}}\,\underline{\underline{s}}\,\underline{\underline{Q}}^T) &= \underline{\underline{Q}}\,\hat{\underline{\underline{s}}}(p,\theta,\underline{\underline{s}})\,\underline{\underline{Q}}^T
\end{aligned}}$$

we can only use invariants of $\underline{\underline{s}}$