Lecture 21: Constitutive Theory

Logistics: – PS8 is due Thursday

Last time: – Frame indifference

- two observers φ, φ^*
- Frame indifference

$$\phi^*(\underline{x}^*, t) = \phi(\underline{x}, t)$$

$$\underline{w}^*(\underline{x}^*, t) = \underline{\underline{Q}}\, \underline{w}(\underline{x}, t)$$

$$\underline{\underline{S}}^*(\underline{x}^*, t) = \underline{\underline{Q}}\, \underline{\underline{S}}(\underline{x}, t)\, \underline{\underline{Q}}^T$$

- Velocity & Acceleration are not frame indifferent

  ⇒ Centrifugal & Coriolis acc.

- Functions of fields

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\, \underline{\underline{S}}\, \underline{\underline{Q}}^T) = \underline{\underline{Q}}\, \hat{\underline{\underline{\sigma}}}(\underline{\underline{S}})\, \underline{\underline{Q}}^T$$

Today: – Isotropic tensor functions

- Fourth order tensors
- Material constraints

Isotropic Functions

Functions that are frameindifferent are also called isotropic. For two ref. frames related by rotation $\underline{\underline{Q}}$ we have

$$\boxed{\begin{array}{lll}
\phi(\theta(\underline{x}^*,t)) = \phi(\theta(\underline{x},t)) & \phi(\underline{\underline{Q}}\,\overset{\underline{x}^*}{\underline{v}}) = \phi(\overset{\underline{x}}{\underline{v}}) & \phi(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \phi(\underline{\underline{S}}) \\
\underline{u}(\theta(\underline{x}^*)) = \underline{\underline{Q}}\,\underline{u}(\theta(\underline{x})) & \underline{u}(\underline{\underline{Q}}\,\underline{v}(\underline{x}^*)) = \underline{\underline{Q}}\,\underline{u}(\underline{v}(\underline{x})) & \underline{u}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{u}(\underline{\underline{S}}) \\
\underline{\underline{\sigma}}(\theta(\underline{x}^*)) = \underline{\underline{Q}}\,\underline{\underline{\sigma}}(\theta(\underline{x}))\,\underline{\underline{Q}}^T & \underline{\underline{\sigma}}(\underline{\underline{Q}}\,\underline{v}(\underline{x}^*)) = \underline{\underline{Q}}\,\underline{\underline{\sigma}}(\underline{v}(\underline{x}))\,\underline{\underline{Q}}^T & \underline{\underline{\sigma}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{\sigma}}\,\underline{\underline{Q}}^T
\end{array}}$$

ϕ = scalar val. fun $\quad$ $\underline{u}$ = vector val. fun $\quad$ $\underline{\underline{\sigma}}$ = tensor val. fun

θ = scalar field $\quad$ $\underline{v}$ = vector field $\quad$ $\underline{\underline{S}}$ = tensor field

Example: 1) $\phi(\underline{\underline{S}}) = \det(\underline{\underline{S}})$

$$\phi(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \cancelto{1}{\det(\underline{\underline{Q}})}\,\det(\underline{\underline{S}})\,\cancelto{1}{\det(\underline{\underline{Q}}^T)}$$

$$= \phi(\underline{\underline{S}})$$

2) $\underline{u}(\underline{v},\underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\underline{u}(\underline{\underline{Q}}\,\underline{v}, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = Q\,A\,\cancelto{I}{Q^T Q}\,v$$

$$= \underline{\underline{Q}}\,\underline{u}(\underline{v},\underline{\underline{A}})$$

Representation of isotropic tensor functions

The most general form of an isotropic tensor function that maps one symmetric tensor to another symm. tensor is given by

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(\mathcal{I}_A)\,\underline{\underline{I}} + \alpha_1(\mathcal{I}_A)\,\underline{\underline{A}} + \alpha_2(\mathcal{I}_A)\,\underline{\underline{A}}^2}$$

Rivlin-Ericksen Representation Theorem

where $\alpha_0, \alpha_1, \alpha_2$ are functions of the set of principal invariants of $\underline{\underline{A}}$

- $\underline{\underline{G}}$ is clearly symm. if $\underline{\underline{A}}$ is symm.
- To see that G is isotropic

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underbrace{QA\cancel{Q^TQ}AQ^T}_{\underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T}$$

$$\underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\underline{\underline{Q}}^T = \alpha_0 \cancel{\underline{\underline{Q}}\,\underline{\underline{I}}\,\underline{\underline{Q}}^T}^{\underline{\underline{I}}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}^2\underline{\underline{Q}}^T \quad \checkmark$$

because coeff. only depend on invariants of $\underline{\underline{A}}$

$\Rightarrow \underline{\underline{G}}(\underline{\underline{A}})$ isotropic!

A second representation can be obtained by eliminating $\underline{\underline{A}}^2$ term using Caley-Hamilton

$$\underline{\underline{A}}^3 - I_1(A)\,\underline{\underline{A}}^2 + I_2(\underline{\underline{A}})\,\underline{\underline{A}} - I_3(\underline{\underline{A}})\,\underline{\underline{I}} = 0$$

multiply $\underline{\underline{A}}^{-1}$

$$\underline{\underline{A}}^2 - I_1\,\underline{\underline{A}} + I_2\,\underline{\underline{I}} - I_3\,A^{-1} = 0$$

$$\underline{\underline{A}}^2 = I_1\,\underline{\underline{A}} - I_2\,\underline{\underline{I}} + I_3\,A^{-1}$$

substitute into $G(A)$

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \beta_0(I_A)\,\underline{\underline{I}} + \beta_1(I_A)\,\underline{\underline{A}} + \beta_2(I_A)\,\underline{\underline{A}}^{-1}}$$

$\beta_0 = \alpha_0 - I_2\,\alpha_2 \quad \beta_1 = \alpha_1 - I_1\alpha_2 \quad \beta_2 = I_3\,\alpha_2$

$\Rightarrow$ used in hyperelastic materials.

Isotropic 4-th order tensors

If $\underset{=}{G}(\underset{=}{A})$ is linear then it can be written

$$\underset{=}{G}(\underset{=}{A}) = \mathbb{C}\,\underset{=}{A}$$

where $\mathbb{C}$ is a 4-th order tensor.

If we also