Lecture 22: Ideal Fluids

Logistics: - PS8 due

- PS5 graded

- Try to post new one

Last time: - Isotropic tensor functions

$$\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\,\underline{\underline{I}} + \alpha_1(I_A)\,\underline{\underline{A}} + \alpha_2(I_A)\,\underline{\underline{A}}^2$$

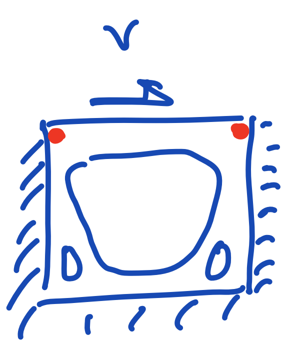
- linear isotropic

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\, tr(\underline{\underline{A}})\,\underline{\underline{I}} + 2\mu\, sym(\underline{\underline{A}})$$

- 4^{th} order tensors

$$\mathbb{C} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l$$

$$C_{ijkl} = \underline{e}_i \cdot \mathbb{C}(\underline{e}_k \otimes \underline{e}_l)\,\underline{e}_j$$



- $\underline{\underline{U}} = \mathbb{C}\,\underline{\underline{A}} \qquad U_{ij} = C_{ijkl}A_{kl}$

- Material constraints $\gamma(\underline{\underline{E}}) = 0$

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a$$

p is multiplier that enforces γ

Today: Ideal Fluids.

Isothermal Fluid Mechanics

→ application of Eulerian balance law

$\nabla_x = \nabla$

→ neglect thermal effects

10 equations

$\underline{v}_m = \dot{\underline{\varphi}}$	3 kinematic
$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$	mass balance
$\rho \dot{\underline{v}} = \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}$	3 lin. mom
$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$	3 ang. mom

16 unknowns: $\underset{3}{\underline{\varphi}}$ $\underset{3}{\underline{v}}$ $\underset{1}{\rho}$ $\underset{9}{\underline{\underline{\sigma}}}$ $= 16$

⇒ Constitutive relation that relates 6 independent comp. $\underline{\underline{\sigma}}$ to $\underline{v}$

Material constraint: add both 1 equ ($\gamma(\underline{\underline{F}}) = 0$) and 1 unknown multiplier q.

Ideal fluids

A fluid is ideal if

1) uniform ref. mass density: $\rho_0(\underline{X}) = \rho_0 > 0$

2) Incompressible: $\nabla \cdot \underline{v} = 0$

3) Cauchy stress is spherical: $\underline{\underline{\sigma}} = -p\underline{\underline{I}}$

$\Rightarrow$ no shear stress $\quad \underline{t} = \underline{\underline{\sigma}}\,\underline{n} = -p\underline{n}$

$1+2 \Rightarrow \rho(\underline{x},t) = \rho_0$

Subst. into mass balance

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \nabla\cdot(\cancel{\rho_0}\underline{v}) = 0 \quad \Rightarrow \quad \boxed{\nabla\cdot\underline{v} = 0}$$

continuity eqn

Subst. into mom. balance

$$\rho_0 \dot{\underline{v}} = -\nabla\cdot(p\underline{\underline{I}}) + \rho_0 \underline{b}$$

expand mat. deriv $\quad \dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla\underline{v})\underline{v}$

$(\underline{v}\cdot\nabla)\underline{v}$

Closed sys. of eqns for $\underline{v}$ & p

$$\boxed{\begin{aligned} \frac{\partial \underline{v}}{\partial t} + \underline{(\nabla\underline{v})\underline{v}} &= -\frac{1}{\rho_0}\nabla p + \underline{b} \\ \nabla\cdot\underline{v} &= 0 \end{aligned}}$$

Euler Equations

4 eqns for 4 unknowns

Note: p has an undetermined constant.

Frame-indifference of Euler's eqns

Stress field in ideal fluid is entirely reactive

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r - \underline{\underline{\sigma}}^a = -p\underline{\underline{I}}$$

$$\underline{\underline{\sigma}}^r = -p\underline{\underline{I}} \qquad \underline{\underline{\sigma}}^a = \underline{\underline{0}}$$

here p is multiplier ass. with incomp. const.

For constrained model we just need to show that $\underline{\underline{\sigma}}^a$ and $\gamma(\underline{\underline{F}}) = 0$ are frame-indifferent.

Assuming a superpose rigid motion

$$\underline{x}^* = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t)$$

- $\gamma(\underline{\underline{F}}^*) = \det(\underline{\underline{F}}^*) - 1 = \det(\underline{\underline{Q}}\,\underline{\underline{F}}) - 1 = \overset{1}{\cancel{\det(\underline{\underline{Q}})}} \det(\underline{\underline{F}}) - 1$
  $= \gamma(\underline{\underline{F}})$ ✓
- $\underline{\underline{\sigma}}^a = \underline{\underline{0}}$ trivially frame-indifferent

$\Rightarrow$ Ideal fluid material model is frame-indif.

Mechanical energy considerations

Even in isothermal model the entropy inequality provides a constraint on material model.

Lecture 18: $\rho_0 \dot{\psi} \le \underline{\underline{\sigma}} : \underline{\underline{d}}$ Mech. Energy Ineq. (MEI)

where $\underline{\underline{\sigma}} : \underline{\underline{d}} = -p \underline{\underline{I}} : \mathrm{sym}(\nabla \underline{v})$

use $\underline{\underline{I}} : \underline{\underline{A}} = \mathrm{tr}(\underline{\underline{A}})$ & $\mathrm{tr}(\mathrm{sym}(A)) = \mathrm{tr}(A)$

$\Rightarrow \quad \underline{\underline{\sigma}} : \underline{\underline{d}} = -p\, \mathrm{tr}(\nabla \underline{v}) = -p\, \nabla \cdot \underline{v} = 0$

In an ideal fluid stress power vanishes

$\dot{\psi} = 0$ Free energy is constant

Steady Bernoulli Streamline Thm

From PS4: $(\nabla \underline{v})\underline{v} = (\nabla \times v) \times \underline{v} + \frac{1}{2} \nabla |\underline{v}|^2$

subst into mom. bal.

$$\frac{\partial \underline{v}}{\partial t} + (\nabla \times \underline{v}) \times \underline{v} = -\frac{1}{2} \nabla |\underline{v}|^2 - \frac{1}{\rho_0} \nabla p + \underline{b}$$

for a conservative body force $\underline{b} = -\nabla \Psi$

where Ψ is the force potential.

collecting ∇ on rhs

$$\boxed{\frac{\partial \underline{v}}{\partial t} + (\nabla \times \underline{v}) \times \underline{v} = -\nabla H}$$

$H = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + \psi$ for gravity $\psi = gz$

H has units of Energy/mass

$E_k = \frac{1}{2} m |\underline{v}|^2 \quad E_G = mgz \quad E_E = m \int_{p_0}^{p} \frac{dp}{\rho} = m\,\frac{p - \cancel{p_0}^{\,0}}{\rho}$

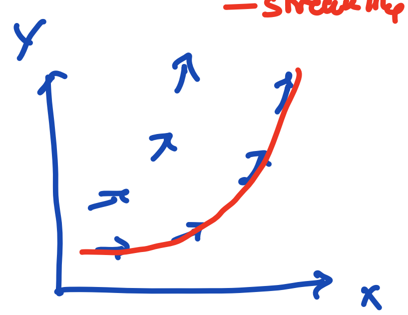
$$\boxed{H = \frac{E}{m} = \frac{E_k}{m} + \frac{E_E}{m} + \frac{E_G}{m} = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + gz}$$

Steady flow

$(\nabla \times \underline{v}) \times \underline{v} = -\nabla H$

take dot product from left

$\cancel{\underline{v} \cdot (\underbrace{\nabla \times \underline{v}}_{\underline{\omega} \perp \underline{v}}) \times \underline{v}}^{\,0} = -\underline{v} \cdot \nabla H$



$\Rightarrow \boxed{\underline{v} \cdot \nabla H = 0}$ Bernulli's Thm for steady flow

implies that H is constant along a streamline

Stream line a curve tangent everywhere to $\underline{v}$

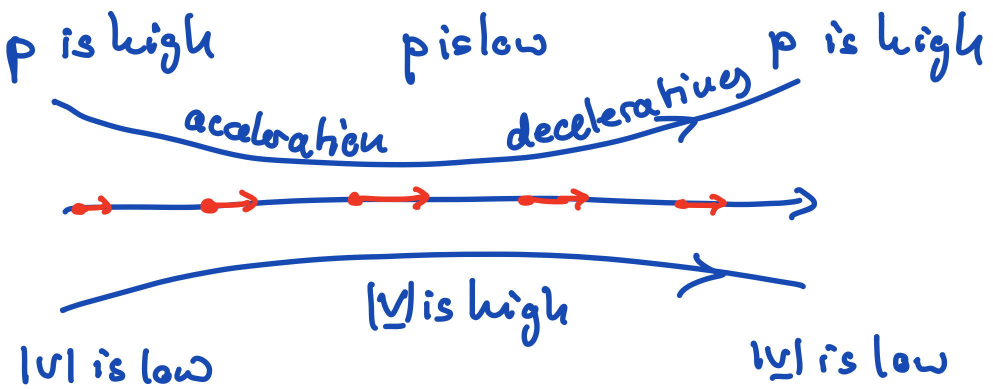
$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

$\Rightarrow$ H is constant along streamline, because energy is constant $\dot{\psi} = 0$

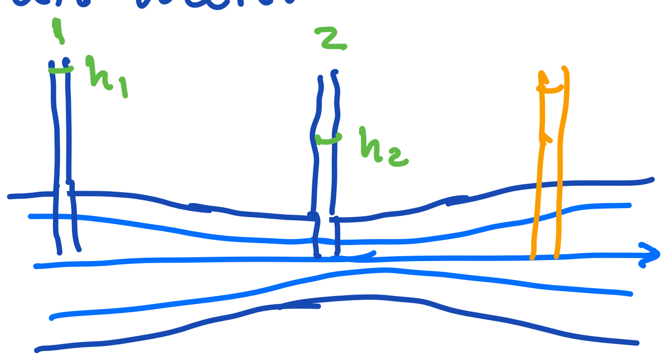
In an ideal fluid there is no energy dissip.

$$H = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + gz = \text{const}$$

if $z = \text{const}$ then increase in $\underline{v}$ requires decrease in p along a streamline



Example: Venturi meter



H is const along along central streamline

$$H = \frac{p_1}{\ } + \frac{1}{2}v_1^2 = \frac{p_2}{\ } + \frac{1}{2}v_2^2 \qquad (z=0)$$

$\rho_0 \qquad \rho_0$

from mass balance: $A_1 v_1 = A_2 v_2 \qquad v_2 = \frac{A_1}{A_2} v_1$

hydrostatics: $p_1 - p_0 = \rho g h_1 \qquad p_2 - p_0 = \rho g h_2$

$\rho g (h_1 - h_2) = \frac{\rho}{2}\left(\frac{A_1^2}{A_2^2} - 1\right) v_1^2$

solve for v_1: $\boxed{v_1^2 = \frac{2g(h_1 - h_2)}{(A_1^2/A_2^2 - 1)}}$

Irrotational Motion

$\underline{v}$ is irrotational if

$$\underline{\underline{W}} = \text{skew}(\nabla \underline{v}) = \underline{\underline{0}} \quad \text{or} \quad \nabla \times \underline{v} = \underline{\omega} = \underline{0}$$

particle experience no net rotation.

Velocity potential

Helmholtz decomposition of velocity

$$\underline{v} = + \nabla \phi + \nabla \times \underline{\psi}$$

for irrotational flow

$$\nabla \times \underline{v} = + \nabla \times \cancelto{0}{\nabla \phi} + \nabla \times \nabla \times \underline{\psi} = \underline{0}$$

$$\Rightarrow \quad \underline{\psi} = \underline{0}$$

Irrotational flows have a scalar velocity potential: $\quad \underline{v} = + \nabla \phi$

$$\Rightarrow \quad \nabla \cdot \underline{v} = + \nabla^2 \phi = 0 \qquad \text{Laplace Eqn}$$

In steady irrotational flow

$$(\cancelto{0}{\nabla \times \underline{v}}) \times \underline{v} = - \nabla H$$

$$\boxed{\nabla H = 0}$$

H is constant in a steady-irrotational ideal fluid

Time dependent irrotational flows

Starting from mom. balance

$$\frac{\partial \cancelto{+\nabla\phi}{\underline{v}}}{\partial t} + (\cancelto{0}{\nabla \times \underline{v}}) \times \underline{v} = - \nabla H$$

$$-\nabla \frac{\partial \phi}{\partial t} + \nabla \left(\frac{1}{2} |\underline{v}|^2 + \frac{P}{\rho_0} + gz \right) = 0$$

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\underline{v}|^2 + \frac{P}{\rho_0} + gz \right) = 0$$

This implies

$$\boxed{\begin{gathered}\frac{\partial\phi}{\partial t}+\frac{1}{2}|\underline{v}|^2+\frac{P}{\rho_0}+gz=0\\ \nabla^2\phi=0\\ \underline{v}=\nabla\phi\end{gathered}}$$

Bernoulli's Thm for irrotational flow

Big simplification but are flows irrotational.

Vorticity equation

vorticity : $\underline{\omega}=\nabla\times\underline{v}$

subst. into mom. bal.

$$\frac{\partial}{\partial t}\underline{v}+\underline{\omega}\times\underline{v}=-\nabla H$$

take the curl

$$\frac{\partial}{\partial t}\underline{\omega}+\nabla\times\underline{\omega}\times\underline{v}=-\nabla\times\nabla H=0$$

where $\nabla\times\underline{\omega}\times\underline{v}=(\nabla\underline{\omega})\underline{v}+\cancel{(\nabla\cdot\underline{v})}^{0}\underline{\omega}-\cancel{(\nabla\cdot\underline{\omega})}^{0}\underline{v}-(\nabla\underline{v})\underline{\omega}$

$$\underbrace{\frac{\partial}{\partial t}\underline{\omega}+(\nabla\underline{\omega})\underline{v}}-(\nabla\underline{v})\underline{\omega}=0$$

$$\boxed{\dot{\underline{\omega}}-(\nabla\underline{v})\underline{\omega}=\underline{0}}$$

Vorticity eqn

Use this to show that an initially irrotational fluid remains irrotational!

Simple proof in 2D:

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} \quad \nabla \underline{v} = \begin{pmatrix} v_{x,x} & v_{x,y} & 0 \\ v_{y,x} & v_{y,y} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \underline{\omega} = \nabla \times \underline{v} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$\Rightarrow \boxed{(\nabla \underline{v})\underline{\omega} = 0 \quad \Rightarrow \quad \dot{\underline{\omega}} = \underline{0}}$$

Vorticity of fluid element is conserved in ideal fluid,

Vorticity is const. along streamlines

In particular if $\underline{\omega} = 0$ everywhere initially it will remain zero.

$\Rightarrow$ Bernoulli's Thm for irrotational flows applicable to broad range of problems.