Lecture 23: Newtonian Fluids

Logistic: – sorry no new HW

Last time: - Ideal fluids

- spherical Cauchy stress: $\underline{\underline{\sigma}} = -p\underline{\underline{I}}$

  Euler Equations

  $$\rho_o \left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v} \right) = -\nabla_x p + \rho_o \underline{b}$$

  $$\nabla_x \cdot \underline{v} = 0$$

- zero stress power → no energy diss.

- Bernoulli theorems

  Steady: $\underline{v} \cdot \nabla_x H = 0$

  $$H = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_o} + gz$$

  Irrotational: $\underline{v} = \nabla_x \phi$

  + steady $\nabla H = 0$

  + transient $\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_o} - gz = f(t)$

  $\nabla_x^2 \phi = 0$

- Vorticity equation

  $$\dot{\underline{\omega}} - (\nabla_x \underline{v})\underline{\omega} = \underline{0}$$

Newtonian Fluid

1) Ref mass density $\rho_0(\underline{X}) = \rho_0$

2) Incompressible $\nabla_x \cdot \underline{v} = 0$

3) Cauchy stress is Newtonian

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mathbb{C}\nabla_x \underline{v}$$

p = pressure

$\mathbb{C}$ 4^{th} order tensor

$\mathbb{C}$ must have left minor sym. $(\mathbb{C}\underline{\underline{A}})^T = \mathbb{C}\underline{\underline{A}}$

$\Rightarrow$ ensures $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$ $\rightarrow$ ang. mom. balance

$\text{tr}(\mathbb{C}\underline{\underline{A}}) = 0$ if $\text{tr}(\underline{\underline{A}}) = 0$

$\Rightarrow$ $p = \frac{1}{3}\text{tr}(\underline{\underline{\sigma}})$ when $\text{tr}(\nabla_x \underline{v}) = \nabla_x \cdot \underline{v} = 0$

Prop. 1 + 2 $\Rightarrow$ $\rho(\underline{x}, t) = \rho_0$

Reactive stress: $\underline{\underline{\sigma}}^r = -p\underline{\underline{I}}$

Active stress: $\underline{\underline{\sigma}}^a = \mathbb{C}\nabla_x \underline{v} = 2\mu \,\text{sym}(\nabla_x \underline{v})$

by frame idifference

μ = abs viscosity

limit $\mu \rightarrow 0$ recover ideal fluid

Navier Stokes Equations

Setting $p = p_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\,\underline{\underline{d}}$

rep. theorem $\alpha_0(I_A) + \alpha_1(I_A)A + \cancel{HOT}$ linear

$\alpha_0 = tr(\sigma)$ $\alpha_1 = 2\mu$ by linearity

substitute into lin. mom. balance

$$\rho_0 \dot{\underline{v}} = \nabla_x \cdot (-p\underline{\underline{I}} + 2\mu\,\underline{\underline{d}}) + \rho_0 \underline{b}$$

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu\,\nabla_x \cdot \nabla_x \underline{v} + \mu\,\nabla_x \cdot (\nabla_x v)^T$$

$$\nabla_x \cdot \nabla_x \underline{v} = v_{i,jj}\,\underline{e}_i = \nabla_x^2 \underline{v}$$

$$\nabla_x \cdot (\nabla_x \underline{v}) = v_{j,ij}\,\underline{e}_i = v_{j,ji}\,\underline{e}_i = \nabla_x (\cancel{\nabla_x \cdot \underline{v}})^{0}$$

$$\Rightarrow \quad \nabla_x \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu\,\nabla_x^2 \underline{v}$$

Navier Stokes Equations

$$\boxed{\begin{gathered} \rho\left[\frac{\partial v}{\partial t} + (\nabla_x \underline{v})\underline{v}\right] = \mu\,\nabla_x^2 \underline{v} - \nabla_x p + \rho_0 b \\ \nabla_x \cdot \underline{v} = 0 \end{gathered}}$$

Frame indifferent

already checked the constraint

$\Rightarrow$ active stress

$$\underline{\underline{\sigma}}^{a} = 2\mu\,\underline{\underline{d}} = \mathbb{C}\,\nabla_x \underline{v}$$

Check left minor symmetry

$$(\mathbb{C}\,\nabla_x \underline{v})^T = (2\mu\,\underline{\underline{d}})^T = 2\mu\,\underline{\underline{d}}^T = 2\mu\,\underline{\underline{d}} = \mathbb{C}\,\nabla_x \underline{v}$$

Trace condition

$$\mathrm{tr}(\mathbb{C}\,\nabla_x \underline{v}) = 2\mu\,\mathrm{tr}(\underline{\underline{d}}) = 0 \quad \text{if} \quad \mathrm{tr}(d) = 0$$

Superposed rigid motion: $\underline{x}^* = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t)$

$$\underline{\underline{\sigma}}^{a*} = \underline{\underline{\sigma}}^{a*}(\underline{x}^*, t) \qquad \underline{\underline{d}}^* = d^*(\underline{x}^*, t)$$

show $\quad \underline{\underline{\sigma}}^{a*} = \underline{\underline{Q}}\,\underline{\underline{\sigma}}^{a}\,\underline{\underline{Q}}^T$

$$\underline{\underline{\sigma}}^{a*} = 2\mu\,\underline{\underline{d}}^* = 2\mu\,\underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T = \underline{\underline{Q}}(2\mu\,\underline{\underline{d}})\underline{\underline{Q}}^T = \underline{\underline{Q}}\,\underline{\underline{\sigma}}^{a}\,\underline{\underline{Q}}^T$$

$\Rightarrow$ using result $d^* = Q\,d\,Q^T$ from Lecture 20.

$\Rightarrow$ Newtonian fluid model is frame indifferent.

Mechanical energy

Dissipation: $\mathcal{D} = \underline{\underline{\sigma}} : \underline{\underline{d}} = (-p\,\underline{\underline{I}} + 2\mu\,\underline{\underline{d}}) : \underline{\underline{d}}$

$$= -p\,\underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla_x \cdot \underline{v} = 0} + 2\mu\,\underline{\underline{d}} : \underline{\underline{d}}$$

Subst. into red. Clausius-Duhem inequality

$$\rho_0 \dot{\psi} \leq 2\mu \, \underline{\underline{d}} : \underline{\underline{d}} = 2\mu \underbrace{|\underline{\underline{d}}|^2}_{\geq 0}$$

$\Rightarrow$ only if $\mu \geq 0$ energy dissipation during flow.

Kinetic energy during Fluid Motion

How (fast) is K dissipated in Ideal & Newtonian flows.

First two results:

1) Integration by parts in fixed Ω with

$\underline{v} = 0 \quad \partial\Omega$

$$\boxed{\int_\Omega (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x = -\int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x}$$

Consider

$$(v_{i,j} v_i)_{,j} = v_{i,jj} v_i + v_{i,j} v_{i,j}$$

$$v_{i,jj} v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$

substitute

$$\int (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x = \int_\Omega \nabla \cdot (\nabla_x \underline{v}^T \underline{v}) \, dV - \int (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV$$

$$= \int_{\partial\Omega} \nabla_x^T v \, \overset{0 \text{ on } \partial\Omega}{\cancel{\underline{v}}} \cdot \underline{n} \, dA - \int (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV$$

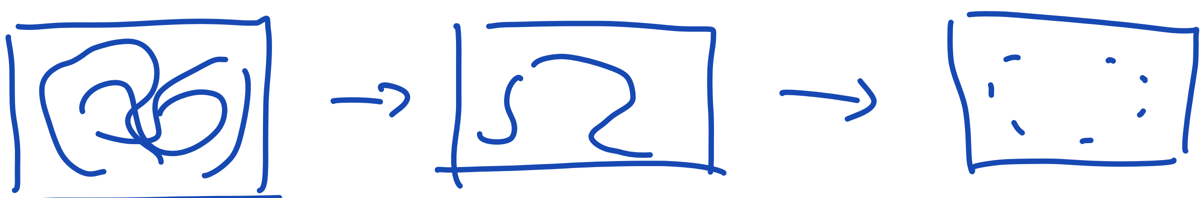
2) Poincaré inequality

$$\|\underline{u}\|_\Omega \leq \lambda \|\nabla_x \underline{u}\|_\Omega \quad \text{for } \underline{u} = 0 \quad \partial\Omega, \lambda > 0$$

$$\boxed{\int_\Omega \overset{\underline{u}\cdot\underline{u}}{|\underline{u}|^2} \, dV_x \leq \lambda \int_\Omega (\nabla_x u) : (\nabla_x \underline{u}) \, dV_x}$$

Units of λ $[L^2]$, constant that scales with area of domain Ω.

For fixed domain Ω with no-slip. BC $\underline{v} = 0$ on $\partial\Omega$ and a conservative force field $\underline{b} = -\nabla_x \varphi$ and kinetic energy $K(t) = \int_\Omega \frac{1}{2} \rho_0 |\underline{v}|^2 dV$ with $K(0) = K_0$



Initial condition with $K_0 > 0$, look at decay.

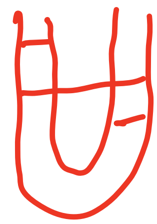
I) for Newtonian fluid

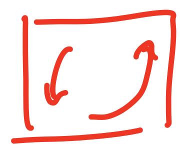
$$\boxed{K(t) \le K_0\, e^{-2\mu t/\lambda\rho_0}}$$

K dissipates exponentially

II) for ideal fluid

$$\boxed{K(t) = K_0}$$





From def of K

$$-\nabla p - \rho g \hat{z} = -\nabla\underbrace{(p - \rho g z)}_{\psi}$$

$$\frac{d}{dt}K(t) = \frac{d}{dt}\int_\Omega \frac{1}{2}\rho_0 |v|^2\, dV_x = \int_\Omega \rho_0\, \dot{v}\cdot v\, dV_x$$

from lin. mom. bal.: $\rho_0 \dot{v} = \mu \nabla_x^2 v - \nabla\psi$

$$\frac{d}{dt}K(t) = \int (\mu \nabla_x^2 v - \nabla\psi)\cdot v\, dV_x$$

show $\int \nabla\phi \cdot v\, dV_x = 0$

$$\nabla_x\cdot(\psi v) = \nabla_x\psi\cdot v + (\cancelto{0}{\nabla_x\cdot v})\,\psi = \nabla_x\psi\cdot v$$

subst and use div thm

$$\frac{d}{dt}K = \int_\Omega \mu\, \nabla_x^2 v\cdot v\, dV - \int_{\partial\Omega} \psi\, \cancelto{0}{v}\cdot n\, dA$$

by integration by parts

$$\frac{dK}{dt} = -\mu\int_\Omega (\nabla_x v):(\nabla_x v)\, dV$$

if fluid is ideal $\mu \to 0$ $\frac{dK}{dt} = 0$ $\to K(t) = K_0$

for Newtonian fluid apply Poincaré

$$\frac{d}{dt} K \leq -\frac{\mu}{\lambda} \int_\Omega |\underline{\omega}|^2 dV_x = -\frac{2\mu}{\rho_0 \lambda} K(t)$$

so that

$$\boxed{\frac{d}{dt} K \leq -\frac{2\mu}{\lambda \rho_0} K}$$

with = an ode for K-decay

by separation of parts

$$K \leq K_0 \, e^{-\frac{2\mu}{\lambda \rho_0} t}$$

λ = geometric factor

$\nu = \frac{\mu}{\rho_0}$ kimatic viscosity

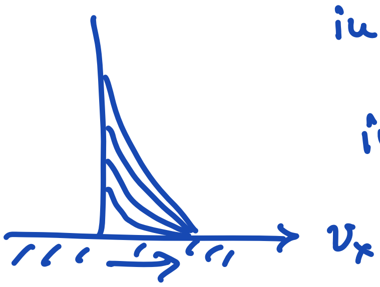
$\mu = [\frac{M}{LT}]$ $\rho_0 = [\frac{M}{L^3}]$ $\nu = \frac{\mu}{\rho} = [\frac{L^2}{T}]$

diffusion coefficient units

$\Rightarrow$ momentum diffusivity

infinite fluid at rest

impulsively sent to move bnd



erf-type soln

Scaling Navier-Stokes equ

$$\rho_0\left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\,\underline{v}\right) = \mu \nabla_x^2 \underline{v} - \nabla_x \pi$$

where $\pi = p + \rho g z$ "reduced pressure" φ

dependent var.: $\underline{v}$, π

independent var.: $\underline{x}$, t

parameters: ρ, μ (ν)

+ geometry, BC, IC

Use parameters to render variable dimensionless

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

subst into lin. mom. bal.

$$\frac{\rho_0 v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho_0 v_c^2}{x_c}(\nabla_x' \underline{v}')\,\underline{v}' = \frac{\mu v_c}{x_c^2}\nabla_x'^2 \underline{v}' - \frac{\pi_c}{x_c}\nabla_x' \pi'$$

have to pick a term to scale to

$\Rightarrow$ pick diff. mom. transport

divide by $\mu v_c / x_c^2$

$$\underbrace{\frac{x_c^2}{\nu t_c}}_{\Pi_1}\frac{\partial \underline{v}'}{\partial t'} + \underbrace{\frac{v_c x_c}{\nu}}_{\Pi_2}(\nabla_x' \underline{v}')\,\underline{v}' = \nabla_x'^2 \underline{v}' - \underbrace{\frac{\pi_c x_c}{\mu v_c}}_{\Pi_3}\nabla_x' \pi'$$

$$\Pi_3 = \frac{\pi_c x_c}{\mu v_c} = 1 \implies \pi_c = \frac{\mu v_c}{x_c}$$

How to pick t_c ? $t_c = t_A = \frac{x_c}{v_c}$ adv. timescale

$$\Pi_1 = \Pi_2 = \frac{v_c x_c}{\nu} = Re \quad \text{Reynolds number}$$

$$\boxed{Re\left(\frac{\partial \underline{v}'}{\partial t'} + (\nabla_x' \underline{v}')\underline{v}'\right) = \nabla_x'^2 \underline{v}' - \nabla_x' \pi'}$$

Dim. less Navier Stokes Equ.

limit $Re \ll 1$

$$\nabla_x'^2 \underline{v}' = \nabla_x' \pi'$$

$$\nabla_x' \cdot \underline{v}' = 0$$

Re dimensionalize

$$\boxed{\begin{gathered} \mu \nabla_x^2 \underline{v} = \nabla_x \pi \\ \nabla_x \cdot \underline{v} = 0 \end{gathered}}$$ Stokes Equ

- linear & instantaneous