Lecture 24: Power-law creep

Logistics: –

Last time: Newtonian fluids

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mathbb{C}\nabla\underline{v} = -p\underline{\underline{I}} + 2\mu\,\underline{\underline{d}}$$

stress-strain relation is linear

$$\rho_0\left[\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v}\right] = -\mu\nabla_x^2\underline{v} - \nabla_x p + \rho\underline{b}$$

$$\nabla_x \cdot \underline{v} = 0$$

Navier Stokes equations (non-linear)

$$Re = \frac{v_c x_c}{\nu} \qquad \nu = \frac{\mu}{\rho}$$

$Re \ll 1 \Rightarrow$ Stokes equation

$$\mu\nabla_x^2\underline{v} = -\nabla p + \rho\underline{b} \quad \rightarrow \text{linear}$$

Today: Nonlinear constitutive laws

Power-law creep $\Rightarrow$ ductile deformation

Power law creep

Earth Science this most important non linear rheology.

Glaciers, Earth mantle

$$\boxed{\dot{\varepsilon} = A\,\sigma^n}$$

σ = stress

$\dot{\varepsilon}$ = strain rate ($\underline{\underline{d}}$)

n = stress exponent

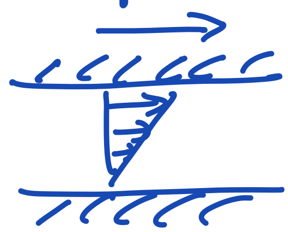
A = pre factor

This is a scalar relation, but we need a tensor form.

$\Rightarrow$ Rheology of the earth, Ranalli

The tensor form can be established from laboratory experiments.

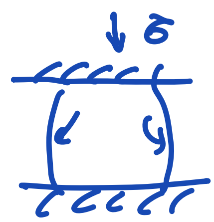
Simple shear



$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{d}} = \underline{\underline{\dot{\varepsilon}}} = \begin{bmatrix} 0 & \dot{\varepsilon}_s & 0 \\ \dot{\varepsilon}_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Uniaxial compression



$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\varepsilon} = \begin{bmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{appox.}$$

$\Rightarrow$ simple configurations with only 1 non-zero component in $\underline{\underline{\sigma}}$ and $\underline{\underline{\dot{\varepsilon}}}$

Suppose simple shear experiments lead to

$$\boxed{\dot{\varepsilon}_s = A\,\sigma_s^n}$$

A is function of T, P, material parameters.

but hat n is constant

Turn into tensorial form

1) Exp. not affected by pressure

$\Rightarrow$ use deviatoric stress $\underline{\underline{\sigma}}'$ & strain rate $\underline{\underline{\dot{\varepsilon}}}'$

2) Frame indifferent $\Rightarrow$ use invariants.

Consider Representation Theorem

$$\dot{\underline{\underline{\varepsilon}}} = \alpha_0(I_{\sigma'})\,\underline{\underline{I}} + \alpha_1(I_{\sigma'})\,\underline{\underline{\sigma}}' + \alpha_2(I_{\sigma'})\,\underline{\underline{\sigma}}'^2$$

$\uparrow$

suppose $n=2$

$$\dot{\varepsilon} = \alpha_2(I_{\sigma'})\,\underline{\underline{\sigma}}'^2 \qquad (Q\sigma Q^T)^2 = Q\sigma \cancel{Q^T Q}^{\,I}\sigma Q^T$$

only works for $n=1$ and $n=2$ but observations show $1 \le n < 5$.

$\Rightarrow$ can't use $\underline{\underline{\sigma}}^n$ directly but power law info has to enter α_1

From lecture 3: Invariants

$$I_1(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 = tr(\underline{\underline{S}}) = S_{11} + S_{22} + S_{33}$$

$$I_2(\underline{\underline{S}}) = \tfrac{1}{2}\left(tr(\underline{\underline{S}})^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

in terms of components

$$-I_2(\underline{\underline{S}}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix}$$

$$= \sigma_{11}\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{33} - \sigma_{13}^2 + \sigma_{22}\sigma_{33} - \sigma_{23}^2$$

$$I_2(\underline{\underline{S}}) = -(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2$$

$$I_3(\underline{\underline{S}}) = \det(\underline{\underline{S}}) = \lambda_1 \lambda_2 \lambda_3$$

$I_1 = 0$ because deviatoric tensors

$$I_3(\dot{\underline{\underline{\varepsilon}}}) \overset{?}{=} \nabla_x \cdot \underline{v} = 0 \quad \text{(must be checked)}$$

$\Rightarrow$ leaves I_2 as only option

Invariants of deviatoric stress:

$I_1(\underline{\underline{\sigma}}') = J_1(\underline{\underline{\sigma}}) = 0$ $\quad$ $J_1(\dot{\underline{\underline{\varepsilon}}}) = 0$ $\quad$ deviatoric

$$I_2(\underline{\underline{\sigma}}') = \boxed{J_2(\underline{\underline{\sigma}}) = \frac{1}{2}\, \underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'}, \quad I_2(\dot{\underline{\underline{\varepsilon}}}') = \boxed{J_2(\dot{\underline{\underline{\varepsilon}}}) = \frac{1}{2}\, \dot{\underline{\underline{\varepsilon}}}' : \dot{\underline{\underline{\varepsilon}}}'}$$

To see this:

$$I_2(\underline{\underline{\sigma}}') = \sigma_{12}'^2 + \sigma_{13}'^2 + \sigma_{23}'^2$$

$$\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}' = \sigma_{ij}'\sigma_{ij}' \overset{?}{=} \sigma_{ij}'^2 = \sigma_{12}'^2 + \sigma_{13}'^2 + \sigma_{23}'^2 + \sigma_{21}'^2 + \sigma_{31}'^2 + \sigma_{32}'^2$$

$$= 2(\sigma_{12}'^2 + \sigma_{13}'^2 + \sigma_{23}'^2) = 2 J_2(\underline{\underline{\sigma}})$$

We can define an effective stress & strain rate

$$\sigma'_E = \sqrt{\tfrac{1}{2}\,\underline{\underline{\sigma}}':\underline{\underline{\sigma}}'} \qquad \dot{\underline{\underline{\varepsilon}}}_E = \dot{\underline{\underline{\varepsilon}}}'_E = \sqrt{\tfrac{1}{2}\,\dot{\underline{\underline{\varepsilon}}}:\dot{\underline{\underline{\varepsilon}}}}$$

power law in terms of effective quantities

$$\boxed{\dot{\varepsilon}_E = A\,\sigma_E^{n}}$$

is in terms of invariants and hence objective

To make this tensorial the Representation

Thm leaves us only one option

$$\boxed{\dot{\underline{\underline{\varepsilon}}} = \alpha_1(I_{\sigma'})\,\underline{\underline{\sigma}}'} + \cancelto{0}{\alpha_2}(I_{\sigma'})\,\underline{\underline{\sigma}}'^2$$

$$\Rightarrow \quad \dot{\varepsilon}_E = \alpha_1(I_{\sigma'})\,\sigma'_E$$

combine and solve for α_1

$$A\,\sigma_E'^{\,n} = \alpha_1(I_{\sigma'})\,\sigma'_E$$

$$\alpha_1(I_{\sigma'}) = A\,\sigma_E'^{\,n-1}$$

Tensor form for power-law creep

$$\boxed{\dot{\underline{\underline{\varepsilon}}} = A\,\sigma_E'^{\,(n-1)}\,\underline{\underline{\sigma}}'}$$

$\Rightarrow$ clearly frame invariant.

Example: Simple shear

$$\underline{\underline{\sigma}} = \begin{pmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{\underline{\sigma}}' \qquad \dot{\underline{\underline{\varepsilon}}} = \dot{\underline{\underline{\varepsilon}}}'$$

$$\sigma_E' = \sqrt{\tfrac{1}{2}\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'} = \sigma_s \qquad \dot{\varepsilon}_E = \dot{\varepsilon}_s$$

substitute $\dot{\underline{\underline{\varepsilon}}} = A\, \sigma_E'^{\,n-1}\, \overset{\sigma_{12}}{\underline{\underline{\sigma}}'} \Rightarrow \dot{\varepsilon}_s = A\, \sigma_E'^{\,(n-1)}\, \sigma_s$

$$= A\, \sigma_s^{\,n-1}\, \sigma_s$$

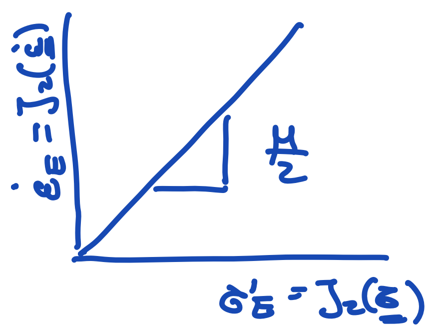
$$= A\, \sigma_s^{\,n}$$

Effective viscosity of power law creep

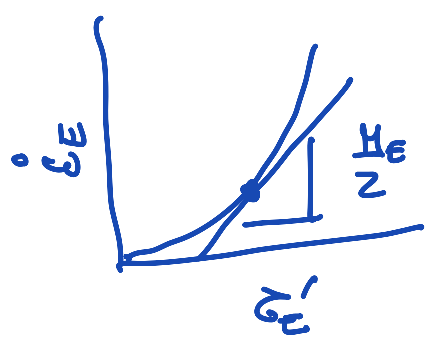
Newtonian: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\, \dot{\underline{\underline{\varepsilon}}}$

$$\underline{\underline{\sigma}}' = 2\mu\, \dot{\underline{\underline{\varepsilon}}}$$

$$\Rightarrow \mu = \frac{|\sigma'|}{2|\dot{\varepsilon}|} = \frac{\sigma_E'}{2\dot{\varepsilon}_E}$$



Non-Newtonian: $\underline{\underline{\sigma}}' = \underbrace{\frac{1}{A}\, \sigma_E'^{\,1-n}}_{2\mu_E(\sigma_E')}\, \dot{\underline{\underline{\varepsilon}}}$



$\Rightarrow$ effective viscosity is function of stress level.