# Lecture 25: Elastic Solids

Logistics: - please fill out class evaluations

Last time: - Power-law creep

$$\dot{\varepsilon}_s = A\,\sigma_s^n \qquad n=1 \text{ Newtonian}$$

- $J_2(\underline{\underline{\sigma}}) = I_2(\underline{\underline{\sigma}}') = \frac{1}{2}\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'$

$\Rightarrow$ effective stress & strain rate: $\sigma_E' = \sqrt{\frac{1}{2}\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'}$

$$\dot{\varepsilon}_E = A\,\sigma_E'^{\,n}$$

- From representation thm

$$\dot{\underline{\underline{\varepsilon}}} = \alpha_1(I_{\sigma'})\,\underline{\underline{\sigma}}' + \cancel{\dots}$$

$$\Rightarrow \boxed{\dot{\underline{\underline{\varepsilon}}} = A\,\sigma_E'^{(n-1)}\,\underline{\underline{\sigma}}'}$$

Today: - Elastic solids

→ Lagrangian formulation

- stress response function
- Elasto dynamic / Elasto static equations
- Material frame-indifference
- Isotropic response functions

# Solid Mechanics

- neglect thermal effects

3 kinematic: $\dot{\underline{v}} = \dot{\underline{\varphi}}$

3 lin. mom: $\rho_0 \dot{\underline{v}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}$

3 ang. mom: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^T$ $\quad \underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$

$\Rightarrow$ 9 equs for following 15 unknown

$\underline{\varphi}$ $\quad \underline{v}$ $\quad \underline{\underline{P}}$ $\qquad 3+3+9 = 15$

$\Rightarrow$ $15 - 9 = 6$ additional constraints

constitutive laws: $\underline{\underline{P}} = \hat{\underline{\underline{P}}}(\underline{\underline{F}})$

material model is independent of $\underline{v}$

eliminate $\dot{v}$ from lin. mom. bal. by subst.

kinematic equs $\Rightarrow$ unknown $\underline{\varphi}$

$$\boxed{\rho_m \ddot{\underline{\varphi}} = \nabla_X \cdot \hat{\underline{\underline{P}}}(\nabla \underline{\varphi}) + \rho_0 \underline{b}}$$ elastodynamic equation

$$\boxed{\nabla_X \cdot \hat{\underline{\underline{P}}}(\nabla \underline{\varphi}) + \rho_0 \underline{b} = 0}$$ elastostatic equation

# General elastic solids

General $\rightarrow$ specific

1) general isotropic elastic materials
2) Hyperelastic materials
3) Linear elastic materials

distribution

A elastic body has

const. law

1) Cauchy stress has form: $\underline{\underline{\sigma}}_m(\underline{X}, t) = \hat{\underline{\underline{\sigma}}}(\underline{F}(\underline{X}, t), \underline{X})$

heterog.

where $\hat{\underline{\underline{\sigma}}}$ is stress response function

stress only depends on present strain
but not strain history.

$\Rightarrow$ generalization of Hooke's law

2) $\hat{\underline{\underline{\sigma}}}(\underline{F}, \underline{X}) = \hat{\underline{\underline{\sigma}}}^T(\underline{F}, \underline{X})$ symmetry

$\Rightarrow$ ang. mom. bal. is automatically satisfied

A body is homogeneous if $\hat{\underline{\underline{\sigma}}}(\underline{F}) \neq \hat{\underline{\underline{\sigma}}}(\underline{F}, \underline{X})$

Example of stress response function:

St. Venant - Kirchhoff model

$$\boxed{\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \lambda \, \mathrm{tr}(\underline{\underline{E}})\, \underline{\underline{I}} + 2\mu \underline{\underline{E}}}$$

form similar to Newtonian fluid

$$\left( \underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu \, \mathrm{sym}(\underline{\underline{d}}) \right)$$

where $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$ Green Lagrange strain tensor

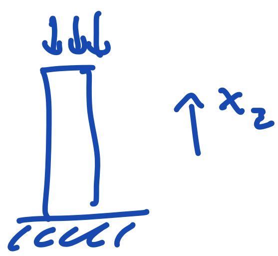
$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ right-Cauchy Green strain tensor

$\lambda, \mu > 0$ scalar material parameters

Example: Uniaxial compression

$$\underline{\varphi} = \begin{pmatrix} X_1 \\ q X_2 \\ X_3 \end{pmatrix} \qquad 0 \le q \le 1$$

compression




$$\underline{\underline{F}} = \nabla \underline{\varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{pmatrix} 1 & & \\ & q^2 & \\ & & 1 \end{pmatrix}$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(q^2-1) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

What are stresses?

$$\underline{\underline{\Sigma}} = \hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \lambda \operatorname{tr} \underline{\underline{E}} + 2\mu \underline{\underline{E}} = \begin{bmatrix} \frac{\lambda}{2}(q^2-1) & & \\ & (\frac{\lambda}{2}+\mu)(q^2-1) & \\ & & \frac{\lambda}{2}(q^2-1) \end{bmatrix}$$

$$\underline{\underline{\Sigma}} = \underline{\underline{F}}^{-1}\underline{\underline{P}} \qquad \underline{\underline{P}} = \underline{\underline{F}}\,\underline{\underline{\Sigma}}$$

$$\underline{\underline{P}} = \begin{bmatrix} \frac{\lambda}{2}(q^2-1) & & \\ & (\frac{\lambda}{2}+\mu)(q^3-q^2) & \\ & & \frac{\lambda}{2}(q^2-1) \end{bmatrix}$$

What is force neccessary for compression?

$$\underline{f}_{\underline{e}_z} = \int_A \underline{\underline{P}}\,\underline{N}\, dA_X = \pm\left(\frac{\lambda}{2}+\mu\right)\left(q^3-q^2\right)\underline{e}_z$$

In limit of extreme compression $q \to 0$ we expect to have to apply an extreme force.

$$\lim_{q\to 0} |f_z| = \left(\frac{\lambda}{2}+\mu\right)\left(q^3-q^2\right) \to 0$$

# Material Frame-Indifference

Cauchy stress is only frame-indifferent if the stress response $\hat{\underline{\underline{\sigma}}}$ has the form

$$\boxed{\begin{aligned} \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) &= \underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{F}}^T \\ \hat{\underline{\underline{P}}}(\underline{\underline{F}}) &= \underline{\underline{F}}\,\bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) \\ \hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) &= \bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) \end{aligned}}$$

where $\bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) = \sqrt{\det(\underline{\underline{C}})}\;\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})$

Implication: $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} = \nabla\varphi^T\nabla\varphi$

$\Rightarrow \underline{\underline{C}}$ is non-lin. function $\varphi$

$\Rightarrow \hat{\sigma}, \hat{\Sigma}, \hat{P}$ are non-linear fun. $\varphi$

$\rho_0\ddot{\varphi} = \nabla_x\cdot\hat{\underline{\underline{P}}}(\nabla\varphi) + \rho_0\underline{b}_m \Rightarrow$ non-linear PDE

Consider superposed rigid motion $\underline{x}^* = \underline{\underline{Q}}(t)\,\underline{x} + \underline{c}(t)$

by frame indifference (Lect. 20)

$\underline{\underline{Q}}^T\underline{\underline{\sigma}}^*\underline{\underline{Q}} = \underline{\underline{\sigma}}$ or $\underline{\underline{Q}}^T\underline{\underline{\sigma}}_m^*\underline{\underline{Q}} = \underline{\underline{\sigma}}_m$

stress field is always given by stress response fun.

$$\underline{\underline{\sigma}}_m = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}(\underline{X},t)) \quad \text{and} \quad \underline{\underline{\sigma}}_m^* = \hat{\sigma}(\underline{\underline{F}}^*(\underline{X},t))$$

note $\hat{\sigma}$ is independent of ref. frame.

frame indifference $\quad \underline{F}^* = Q\,\underline{F}$

$$\boxed{\underline{Q}^T \hat{\underline{\sigma}}(\underline{\underline{Q}}\,\underline{\underline{F}})\,\underline{\underline{Q}} = \hat{\sigma}(\underline{\underline{F}})}$$

Polar decomp. $\quad \underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{Q}}^T \underline{\underline{U}}$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{R}}\;\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{Q}}^T\underline{\underline{U}})\,\underline{\underline{R}}^T = \underline{\underline{R}}\,\hat{\sigma}(\underline{\underline{U}})\,\underline{\underline{R}}^T$$

Define $\underline{\underline{C}}^{1/2} = \sqrt{\underline{\underline{C}}} \qquad \underline{\underline{C}}^{-1/2} = (\sqrt{\underline{\underline{C}}})^{-1}$ so that $\underline{\underline{U}} = \underline{\underline{C}}^{1/2}$

and $\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{C}}^{-1/2}$ substitute

$$\underline{\hat{\underline{\sigma}}(\underline{\underline{F}}) = \underline{\underline{F}}\;\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,F^T}, \qquad \bar{\underline{\underline{\sigma}}} = C^{-1/2}\,\hat{\sigma}(C^{1/2})\,C^{-1/2} \quad \checkmark$$

Example : St Venant Kirchhoff

$$\hat{\underline{\underline{\Sigma}}} = \lambda\,\mathrm{tr}(\underline{\underline{E}}) + 2\mu\,\underline{\underline{E}} \qquad\qquad E = \tfrac{1}{2}(F^TF - I)$$

$$\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) = \frac{\lambda}{2}\,\mathrm{tr}(\underline{\underline{C}} - \underline{\underline{I}})\,\underline{\underline{I}} + \frac{\mu}{2}(\underline{\underline{C}} - \underline{\underline{I}})$$

$$\nabla\varphi^T\nabla\varphi$$

## Isotropic stress response

A body is isotropic if

$$\boxed{\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}}^T)}$$

$\Rightarrow$ material has same stiffness in every direction.

To get isotropic stress response we need to relate concept of isotropic material to isotropic tensor function → frame-indiff.

$$\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{Q}}^T \quad \text{and} \quad \bar{\underline{\underline{\Sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}})\,\underline{\underline{Q}}^T$$

Frame indif stress resp: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underbrace{\underline{\underline{F}}^T\underline{\underline{F}}}_{\underline{\underline{C}}})\,\underline{\underline{F}}^T$

Isotropic material: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}}^T)$

$$\underline{\underline{F}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underbrace{\underline{\underline{F}}^T\,\underline{\underline{F}}}_{\underline{\underline{C}}}\,\underline{\underline{Q}}^T)\,\underline{\underline{Q}}\,\underline{\underline{F}}^T$$

$$\underline{\underline{F}}\,\boxed{\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\boxed{\underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T)\,\underline{\underline{Q}}}\,\underline{\underline{F}}^T$$

$$\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}}) = \underline{\underline{Q}}^T\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T)\,\underline{\underline{Q}}$$

$$\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{Q}}^T \quad \checkmark$$

$\Rightarrow$ for isotropic material $\bar{\underline{\underline{\sigma}}}$ is isotropic tensor function

$\Rightarrow$ use representation Thm for isotropic tensor functions

For an isotropic body the stress response $\hat{\underline{\underline{\sigma}}}$ is frame-indifferent only if written as

$$\boxed{\begin{aligned}
\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) &= \underline{\underline{F}}\left[\beta_0(\mathcal{I}_c)\underline{\underline{I}} + \beta_1(\mathcal{I}_c)\underline{\underline{C}} + \beta_2(\mathcal{I}_c)\underline{\underline{C}}^{-1}\right]\underline{\underline{F}}^T \\
\hat{\underline{\underline{P}}}(\underline{\underline{F}}) &= \underline{\underline{F}}\left[\gamma_0(\mathcal{I}_c)\underline{\underline{I}} + \gamma_1(\mathcal{I}_c)\underline{\underline{C}} + \gamma_2(\mathcal{I}_c)\underline{\underline{C}}^{-1}\right] \\
\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) &= \gamma_0(\mathcal{I}_c)\underline{\underline{I}} + \gamma_1(\mathcal{I}_c)\underline{\underline{C}} + \gamma_2(\mathcal{I}_c)\underline{\underline{C}}^{-1}
\end{aligned}}$$

follows from $\hat{\underline{\underline{\sigma}}} = \underline{\underline{F}}\,\bar{\sigma}(C)\,\underline{\underline{F}}^T$ and

resp. theorem $\bar{\sigma}(\underline{\underline{C}}) = \beta_0 I + \beta_1 \underline{\underline{C}} + \beta_2 \underline{\underline{C}}^{-1}$

where $\gamma_i = \beta_i \sqrt{\det(\underline{\underline{C}})}$

Example: St. Venant-Kirchhoff

$$\underline{\underline{\Sigma}}(\underline{\underline{C}}) = \frac{\lambda}{2} \operatorname{tr}(\underline{\underline{C}} - \underline{\underline{I}})\, \underline{\underline{I}} + \mu (\underline{\underline{C}} - \underline{\underline{I}})$$

$$= \underbrace{\left(\frac{\lambda}{2} \operatorname{tr} \underline{\underline{C}} - \frac{3\lambda}{2} - \mu\right)}_{\gamma_0} \underline{\underline{I}} + \underbrace{\mu}_{\gamma_1} \underline{\underline{C}}$$

$\gamma$ $\qquad$ $\gamma_0$ $\qquad$ $\gamma_1$ $\qquad$ $\gamma_2 = 0$