Lecture 26: Hyperelastic solids

Logistics: - please complete missing problem sets ✗

- please fill out course evaluation ✓

Last time: - Elastic solids

- $\rho_0 \ddot{\varphi} = \nabla_X \cdot \hat{\underline{P}}(\nabla \varphi) + \rho_0 \underline{b}_m$

- $\hat{\underline{\underline{\sigma}}}_m = \hat{\underline{\underline{\sigma}}}_m(F(X,t))$ not fun. of history

$\underline{\underline{\sigma}}_m^T = \underline{\underline{\sigma}}_m$ ✓

- Frame Indifference:

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{F}}\, \bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\, \underline{\underline{F}}^T$$

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = \underline{\underline{F}}\, \bar{\underline{\underline{P}}}(\underline{\underline{C}})$$

$$\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = \bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}})$$

$$\Rightarrow \underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}} = \nabla\varphi^T \nabla\varphi$$

intrinsically non-linear in φ

- Isotropic material: $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}\,\underline{\underline{Q}})$

$$\Rightarrow \boxed{\bar{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{C}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\bar{\underline{\underline{\sigma}}}(\underline{\underline{C}})\,\underline{\underline{Q}}^T}$$

- $\bar{\underline{\underline{\sigma}}} = \beta_0(I_c)\, \underline{\underline{I}} + \gamma_1(I_c)\, \underline{\underline{C}} + \gamma_2(I_c)\, \underline{\underline{C}}^{-1}$ Rep. Thm

Hyperelastic Materials

A solid is hyperelastic if:

1) homog. stress response: $\underline{\underline{\sigma}}_m(\underline{X},t) = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}(\underline{X},t))$

2) Strain energy density $W(\underline{\underline{F}})$ scalar-valued tens. fun.

$$\boxed{\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = DW(\underline{\underline{F}})}$$

Lecture 5: $DW(\underline{\underline{F}}) = \frac{\partial W}{\partial F_{ij}} \underline{e}_i \otimes \underline{e}_j$

3) W has property

$$DW(\underline{\underline{F}})\, \underline{\underline{F}}^T = \underline{\underline{F}}\, DW(\underline{\underline{F}})^T$$

$$\hat{\underline{\underline{P}}}\, \underline{\underline{F}}^T = \underline{\underline{F}}\, \hat{\underline{\underline{P}}}^T \Rightarrow \hat{\underline{\underline{\Sigma}}}^T = \hat{\underline{\underline{\Sigma}}}$$

$\Rightarrow$ satisfies ang. mom. bal.

Frame - Indifference

$\Rightarrow$ W be of the form

$$\boxed{\begin{aligned} &W(\underline{\underline{F}}) = \overline{W}(\underline{\underline{C}}) \qquad \underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}} \\ &\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}}) = 2\, D\overline{W}(\underline{\underline{C}}) \\ &\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = 2\, \underline{\underline{F}}\, D\overline{W}(\underline{\underline{C}}) \\ &\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = 2 \det(C)^{-1/2}\, F\, D\overline{W}(C)\, F^T \end{aligned}}$$

$\Rightarrow$ W need to be written in terms of $\underline{\underline{C}}$

Show $\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = DW(\underline{\underline{F}})$ and $W(\underline{\underline{F}}) = \bar{W}(\underline{\underline{C}})$

implies $\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = 2\, \underline{\underline{F}}\, D\bar{W}(\underline{\underline{C}})$:

$$\hat{P}_{ij}(\underline{\underline{F}}) = \frac{\partial W}{\partial F_{ij}}(\underline{\underline{F}}) = \frac{\partial \bar{W}}{\partial C_{mL}} \frac{\partial C_{mL}}{\partial F_{ij}} \qquad \text{since } C_{mL} = F_{km} F_{kL}$$

so that
$$\frac{\partial C_{mL}}{\partial F_{ij}} = \frac{\partial}{\partial F_{ij}}\left(F_{km} F_{kL}\right) = F_{km} \frac{\partial F_{kL}}{\partial F_{ij}} + \frac{\partial F_{km}}{\partial F_{ij}} F_{kL}$$
$$= F_{km}\, \delta_{ki}\, \delta_{Lj} + F_{kL}\, \delta_{ki}\, \delta_{mj}$$
$$= F_{im}\, \delta_{Lj} + F_{iL}\, \delta_{mj}$$

substitute

$$\hat{P}_{ij} = \frac{\partial \bar{W}}{\partial C_{mL}}\left(F_{im}\, \delta_{Lj} + F_{iL}\, \delta_{mj}\right)$$
$$= \frac{\partial \bar{W}}{\partial C_{mj}} F_{im} + F_{iL} \frac{\partial \bar{W}}{\partial C_{jL}} \qquad C_{jL} = C_{Lj}$$

$$= 2\left(F_{im} \frac{\partial \bar{W}}{\partial C_{mj}}\right)$$

$\Rightarrow$ $\hat{\underline{\underline{P}}} = 2\, \underline{\underline{F}}\, D\bar{W}(\underline{\underline{C}})$ ✓

From def of 1st Piola-Kirchhoff stress

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = \det(\underline{\underline{F}})\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})\,\underline{\underline{F}}^{-T} = \det(C)^{1/2}\,\hat{\sigma}(F)\,F^{-T}$$

has to be equal to above

$$\hat{P} = 2\,F\,D\bar{W}(\underline{\underline{C}}) = \det(\underline{\underline{C}})^{1/2}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})\,\underline{\underline{F}}^{-T}$$

solve for $\hat{\underline{\underline{\sigma}}}$

$$\Rightarrow \quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}}) = \underline{\underline{F}}\,\underbrace{\left[2\det(\underline{\underline{C}})^{-1/2}\,D\bar{W}(\underline{\underline{C}})\right]}\,\underline{\underline{F}}^{T}$$

$$\underline{\underline{F}} \quad \bar{\underline{\underline{\sigma}}}(C) \quad \underline{\underline{F}}^{T}$$

hence $\hat{\sigma}$ is frame-indifferent

Example: St. Venant-Kirchhoff

$$\bar{\underline{\underline{\Sigma}}}(\underline{\underline{C}}) = \frac{\lambda}{2}\left[\mathrm{tr}(\underline{\underline{C}}-\underline{\underline{I}})\right]\underline{\underline{I}} + \mu\,(\underline{\underline{C}}-\underline{\underline{I}})$$

The corresponding strain energy density

$$\bar{W}(\underline{\underline{C}}) = \frac{\lambda}{8}\left[\mathrm{tr}(\underline{\underline{C}}-\underline{\underline{I}})\right]^2 + \frac{\mu}{4}\,\mathrm{tr}\left[(\underline{\underline{C}}-\underline{\underline{I}})^2\right] \qquad \text{scalar}$$

Check by taking the derivative $\qquad \underline{\underline{A}}\,\underline{\underline{B}} = A_{ik}B_{kj}\,e_i e_j$

$$\bar{W} = \frac{\lambda}{8}(C_{ii}-\delta_{ii})^2 + \frac{\mu}{4}(C_{ik}-I_{ik})(C_{ki}-I_{ki})$$

$$= \frac{\lambda}{8}(C_{ii}-\delta_{ii})^2 + \frac{\mu}{4}(C_{ik}-I_{ik})^2 \qquad I_{ik} = \delta_{ik}$$

$$\frac{\partial \bar{W}}{\partial C_{lm}} = \frac{\lambda}{4}(C_{ii}-\delta_{ii})\frac{\partial C_{jj}}{\partial C_{lm}} + \frac{\mu}{2}(C_{ik}-\delta_{ik})\frac{\partial C_{ik}}{\partial C_{lm}}$$

$$= \frac{\lambda}{4}\operatorname{tr}(\underline{\underline{C}}-\underline{\underline{I}})\,\underbrace{\delta_{il}\,\delta_{im}} + \frac{\mu}{2}(C_{ik}-\delta_{ik})\,\delta_{il}\,\delta_{km}$$

$$= \frac{\lambda}{4}\operatorname{tr}(\underline{\underline{C}}-\underline{\underline{I}})\,\delta_{lm} + \frac{\mu}{2}(C_{lm}-\delta_{lm})$$

$$D\bar{W}(C) = \frac{\partial \bar{W}}{\partial C_{lm}}\,\underline{e}_l\otimes\underline{e}_m = \frac{\lambda}{4}\operatorname{tr}(\underline{\underline{C}}-\underline{\underline{I}})\,\underline{\underline{I}} + \frac{\mu}{2}(\underline{\underline{C}}-\underline{\underline{I}})$$

so that $\underline{\underline{\Sigma}}(\underline{\underline{C}}) = 2\,D\bar{W}(\underline{\underline{C}})$ ✓

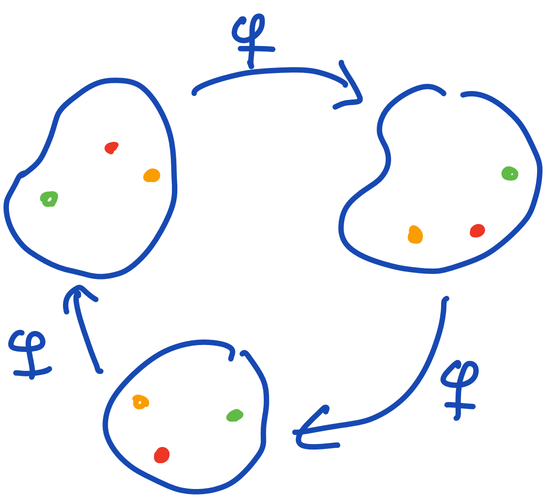
Mechanical energy considerations

→ skipped this in constitutive theory (Lecture 21)

Def: A thermomechanical process is closed

in an interval $[t_0 \;\; t_1]$ if:

$$\varphi(\underline{X},t_0) = \varphi(\underline{X},t_1),\quad \dot{\varphi}(\underline{X},t_0) = \dot{\varphi}(\underline{X},t_1),\quad \Theta(\underline{X},t_1) = \Theta(\underline{X},t_0)$$



thermo: closed system can exchange mech. work with exterior but no ~~he~~ heat.

Def: A body is energetically passive if for any closed process the the material free energy satisfies:

$$\psi(\underline{X}, t_1) - \psi(\underline{X}, t_0) \geq 0$$

If ψ is only a function of current state and not of history then $\psi(\underline{X}, t_1) = \psi(\underline{X}, t_0)$ for any closed process.

Reduced Clausius-Duhem ineq. for an isothermal body $\Theta = \Theta^{\text{const}}$ $\nabla\Theta = 0$

$$\rho_0 \dot{\psi} \leq \underline{\underline{P}} : \dot{\underline{\underline{F}}}$$

integrate ~~of~~ over time interval

$$\underbrace{\rho_0 \left(\psi(\underline{X}, t_1) - \psi(\underline{X}, t_0)\right)}_{\geq 0} \leq \int_{t_0}^{t_1} \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dt$$

for any closed process in an energetically passive material

$\Rightarrow$ Mechanical Energy Inequality (MEI)

$$\boxed{\int_{t_0}^{t_1} \underline{\underline{P}} : \dot{\underline{\underline{F}}}\, dt \geq 0}$$ constraint from 2^{nd} law

MEI in hyperelastic solids

An elastic body satisfies the MEI only if it is hyper elastic: $\hat{\underline{\underline{P}}}(F) = DW(\underline{\underline{F}})$

Consider stress power:

$$\underline{\underline{P}}(\underline{X},t) : \dot{\underline{\underline{F}}}(\underline{X},t) = \hat{\underline{\underline{P}}}(\underline{\underline{F}}(\underline{X},t)) : \dot{\underline{\underline{F}}}$$

$$= DW(F) : \dot{\underline{\underline{F}}}$$

$$= \frac{\partial}{\partial t} W(\underline{\underline{F}}(\underline{X},t))$$ Lecture 5

$\Rightarrow$ stress power is time derivative of strain energy

$\mathcal{P} = \frac{dW}{dt}$ power is rate of work/energy

Integrating stress power

$$\int_{t_0}^{t} \underline{\underline{P}} : \dot{\underline{\underline{F}}}\, dt = \int_{t_0}^{t} \frac{\partial W}{\partial t}\, dt = W(\underline{\underline{F}}(\underline{X},t_1)) - W(\underline{\underline{F}}(X,t_0))$$

$\Rightarrow$ $W = \rho\,\psi$ Energy for volume if process is closed $\underline{F}(X_1, t_1) = \underline{F}(\underline{X}, t_0)$

$\Rightarrow W(t_1) = W(t_0) \Rightarrow \int_{t_0}^{t_1} \underline{P} : \dot{\underline{F}}\, dt = 0$

for any hyperelastic material.

Common hyperelastic models

1) St.Venant-Kirchhoff model

$$\overline{W}(\underline{\underline{C}}) = \frac{\lambda}{2} \operatorname{tr}(\underline{\underline{E}})^2 + \mu \operatorname{tr}(\underline{\underline{E}}^2)$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}}) \qquad \lambda, \mu > 0$$

2) Neo-Hookean model

$$\overline{W}(\underline{\underline{C}}) = a \operatorname{tr}(\underline{\underline{C}}) + \Gamma\left(\sqrt{\det(\underline{\underline{C}})}\right)$$

$$\Gamma(s) = c s^2 + d \ln(s) \qquad a, c, d > 0$$

3) Mooney-Rivlin model

$$\overline{W}(\underline{\underline{C}}) = a \operatorname{tr}(\underline{\underline{C}}) + b \operatorname{tr} \underline{\underline{C}}_* + \Gamma\left(\sqrt{\det(\underline{\underline{C}})}\right)$$

$$\underline{\underline{C}}_* = \det(\underline{\underline{C}})\, \underline{\underline{C}}^{-1}$$

4) Odgen model

$$\overline{W}(\underline{\underline{C}}) = \sum_{i=1}^{M} a_i \operatorname{tr}(\underline{\underline{C}}^{\gamma_i/2}) + \sum_{j=1}^{N} b_j \operatorname{tr}(\underline{\underline{C}}^{\delta_j/2}) + \Gamma\left(\sqrt{\det(\underline{\underline{C}})}\right)$$

where $M, N \geq 1$ $\quad a_i, b_j > 0$ $\quad \gamma_i, \delta_j > 0$

Models 2–4 are standard models for large deformation.
In incompressible case $\Gamma(s)$ is dropped and replaced by constraint $\det(F) = \det(C) = 1$ and corresponding multiplier.
In compressible case $\Gamma(s)$ that ensures that extreme compression requires extreme stresses.