Lecture 27: Linear Elasticity

Logistics: - please fill out course evaluations

Last time: Hyperelastic materials

$$\underline{\underline{\hat{P}}}(\underline{\underline{F}}) = DW(\underline{\underline{F}})$$

Material frame Indiff

$$\hat{P}(\underline{\underline{F}}) = 2\,\underline{\underline{F}}D\overline{W}(\underline{\underline{C}})$$

$\Rightarrow$ inherently nonlinear

- Mechanical energy inequality (2^{nd} law)

  closed system.

  energetically passive

- Common Model

  - Neo-Hookean
  - Mooney-Rivlin
  - Odgen

Today: $\rightarrow$ linearize elastodynamic eqns

Linear Elasticity

Initial boundary value problem

PDE: $\rho_0 \ddot{\varphi} = \nabla_X \cdot \hat{\underline{\underline{P}}}(\underline{\underline{F}}) + \rho_0 \underline{b}_m \qquad X \in \Omega \times [0,T]$

BC: $\varphi = g \qquad \partial\Omega_d$

$\hat{\underline{\underline{P}}}(\underline{\underline{F}})\underline{N} = \underline{h} \qquad \partial\Omega_g$

IC: $\varphi = \underline{X} \qquad \Omega$

$\dot{\varphi} = \underline{V}_0 \qquad \Omega$

Consider a stress free initial condition

at $t=0$ $\quad \underline{\underline{F}} = \underline{\underline{I}} \Rightarrow \hat{\underline{\underline{P}}}(\underline{\underline{I}}) = \hat{\underline{\underline{\Sigma}}}(\underline{\underline{I}}) = \hat{\underline{\underline{S}}}(\underline{\underline{I}}) = \underline{\underline{0}}$

If forcings are small

$|\underline{b}_m| = O(\varepsilon) \quad |g - \underline{X}| = O(\varepsilon) \quad |\underline{h}| = O(\varepsilon) \quad |\underline{V}_0| = O(\varepsilon)$

where $\quad 0 \leq \varepsilon \ll 1$

we expect that the displacement is small

$$|u(\underline{X},t)| = |\varphi(\underline{X},t) - \underline{X}| = O(\varepsilon)$$

$\Rightarrow$ assuming well-posed system

Linearized equations

Express forcings

$\underline{b}_m^\epsilon = \epsilon \underline{b}_m \qquad g^\epsilon = \underline{X} - \epsilon g \qquad \underline{h}^\epsilon = \epsilon \underline{h} \qquad \underline{V}_0^\epsilon = \epsilon \underline{V}_0$

then $\quad \varphi^\epsilon = \underline{X} + \epsilon u + \mathcal{O}(\epsilon^2) \qquad \underline{u}^\epsilon = \epsilon \underline{u} = \varphi^\epsilon - \underline{X}$

def
~~disp.~~ grad.: $\underline{\underline{F}}^\epsilon = \nabla_X \varphi^\epsilon = \underline{\underline{I}} + \epsilon \nabla \underline{u}$

substitute into PDE

$$\rho_0 \frac{\partial^2}{\partial t^2}(\underline{X} + \underline{u}^\epsilon) = \nabla_X \cdot [\hat{\underline{P}}(\underline{\underline{F}}^\epsilon)] + \rho_0 \underline{b}_m^\epsilon$$

$$\rho_0 \ddot{\underline{u}}^\epsilon = \nabla_X \cdot [\hat{\underline{P}}(\underline{\underline{F}}^\epsilon)] + \rho_0 \underline{b}_m^\epsilon$$

Need to deal with $\nabla_X \cdot \hat{\underline{P}}(\underline{\underline{F}}^\epsilon)$

Introduce 3 4^{th} order tensors:

$$A_{ijkl} = \frac{\partial \hat{P}_{ij}}{\partial F_{kl}}(\underline{\underline{I}}) \qquad B_{ijkl} = \frac{\partial \hat{\Sigma}_{ij}}{\partial F_{kl}}(\underline{\underline{I}}) \qquad C_{ijkl} = \frac{\partial \hat{\sigma}_{ij}}{\partial F_{kl}}(\underline{\underline{I}})$$

in tensor notation $H = \nabla_X \underline{u}$

$$\mathbb{A}\underline{\underline{H}} = \frac{d}{d\epsilon} \hat{\underline{P}}(\underline{\underline{I}} + \epsilon \underline{\underline{H}})\Big|_{\epsilon=0} = D\hat{\underline{P}}(\underline{\underline{I}})H \qquad \mathbb{A} = D\hat{\underline{\underline{P}}}(\underline{\underline{I}})$$

$$\mathbb{B}H = D\hat{\underline{\underline{\Sigma}}}(\underline{\underline{I}})\underline{\underline{H}}$$

$$\mathbb{C}H = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})\underline{\underline{H}}$$

Express stress response in Taylor series

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}^{\epsilon}) = \hat{\underline{\underline{P}}}(\underline{\underline{I}} + \epsilon \underline{\underline{H}}) = \cancel{\hat{\underline{\underline{P}}}(\underline{\underline{I}})}^{\,0} + \epsilon \mathbb{A}\, \underline{\underline{H}} + O(\epsilon^2)$$

$$= \epsilon \mathbb{A}\, \underline{\underline{H}}$$

substitute into lin. mom. bal. $\underline{u}^{\epsilon} = \epsilon \underline{u}$ $\quad \underline{b}_m^{\epsilon} = \epsilon \underline{b}_m$

$$\rho_0 \cancel{\epsilon}\, \ddot{\underline{u}} = \cancel{\epsilon}\, \nabla_X \cdot [\mathbb{A} \nabla \underline{u}] + \rho_0 \cancel{\epsilon}\, b_m$$

linearized $\overset{\text{lin.}}{\text{mom.}}$ balance

$$\boxed{\rho_0\, \ddot{\underline{u}} = \nabla_X \cdot [\mathbb{A} \nabla \underline{u}] + \rho_0 b_m}$$ lin. elastodyn. equations

$$\nabla_X \cdot \hat{\underline{\underline{P}}}(\underline{\underline{F}})$$

$$F\bar{P}(\underline{\underline{C}})$$

In lin. case, $|\underline{\varphi} - \underline{X}| = O(\epsilon)$ $\quad x \sim X$

$\Rightarrow$ dont need to distinguish material & current reference frames.

if $\underline{\ddot{u}} = 0$ $\Rightarrow$ elasto static eqns

$$\nabla \cdot [\mathbb{A} \nabla \underline{u}] + \rho_0 b_m = 0$$

Elasticity Tensor

Introduced 4th tensors:

$$\mathbb{A} = D\hat{\underline{\underline{P}}}(\underline{\underline{I}}) \qquad \mathbb{B} = D\hat{\underline{\underline{\Sigma}}}(\underline{\underline{I}}) \qquad \mathbb{C} = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})$$

If initial cond. is stressfree $\Rightarrow \mathbb{A} = \mathbb{B} = \mathbb{C}$

Example: $\mathbb{A} = \mathbb{C}$

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = \det(\underline{\underline{F}})\, \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})\, \underline{\underline{F}}^{-T}$$

differentiate both sides at $\underline{\underline{I}}$

$$D\hat{P}(I)\underline{\underline{H}} = \mathbb{A}\underline{\underline{H}} = \frac{d}{d\varepsilon}\left[\det(I + \epsilon\underline{\underline{H}})\, \hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\underline{\underline{H}})\, (\underline{\underline{I}} + \epsilon\underline{\underline{H}})^{-T}\right]\Big|_{\epsilon=0}$$

$$= \frac{d}{d\varepsilon}\left[\det(\underline{\underline{I}} + \epsilon\underline{\underline{H}})\right]_{\epsilon=0} \cancelto{0}{\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})}\, \underline{\underline{I}}^{-T}$$

$$+ \cancelto{1}{\det(\underline{\underline{I}})}\, \frac{d}{d\varepsilon}\left[\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\underline{\underline{H}})\right]_{\epsilon=0} I^{-T}$$

$$+ \det(I)\, \cancelto{0}{\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})}\, \frac{d}{d\varepsilon}(I + \epsilon H)^{-T}$$

where $\det(I) = 1$ and stress free IC $\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}}) = 0$

$$D\hat{\underline{\underline{P}}}(\underline{\underline{I}})\,\underline{\underline{H}} = \mathbb{A}\,\underline{\underline{H}} = \frac{d}{d\varepsilon}\,\hat{\sigma}(I + \epsilon H)\Big|_{\epsilon=0} = D\hat{\underline{\underline{\sigma}}}(I)\,\underline{\underline{H}} = \mathbb{C}\,\underline{\underline{H}}$$

$$\Rightarrow \mathbb{A} = \mathbb{C}$$

$\Rightarrow$ elastic solid with stressfree IC has a unique elasticity tensor $\mathbb{C}$ which can be determined from any stress response function $\hat{\underline{P}}(\underline{F})$ $\hat{\underline{\Sigma}}(\underline{F})$ $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})$ by linearizing around $\underline{\underline{I}}$.

Balance of angular momentum

$$\hat{\underline{\underline{\sigma}}}(\underline{F})^T = \hat{\underline{\underline{\sigma}}}(\underline{F})$$

$$[\mathbb{C}\,\underline{\underline{H}}]^T = \left(\frac{d}{d\epsilon}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\,\underline{\underline{H}})\Big|_{\epsilon=0}\right)^T$$

$\epsilon \ll 1$

$$= \frac{d}{d\epsilon}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\,\underline{\underline{H}}^T)\Big|_{\epsilon=0} \qquad \text{note} \quad \det(I + \epsilon H) \approx 1$$

$$= \frac{d}{d\epsilon}\,\hat{\underline{\underline{\sigma}}}(I + \epsilon\,\underline{H})\Big|_{\epsilon=0} = \mathbb{C}\,\underline{\underline{H}}$$

Implies that $\mathbb{C}$ has left minor symmetry

$$C_{ijkl} = C_{jikl} \quad \text{or} \quad \underline{\underline{A}} : \mathbb{C}\,\underline{B} = \operatorname{sym}(\underline{\underline{A}}) : \mathbb{C}\,\underline{\underline{B}}$$

Frame-indifference. for isotropic solid

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\;\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})\,\underline{\underline{Q}}^T \qquad \text{taking} \quad \underline{\underline{F}} = \underline{\underline{I}}, \quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{I}}) = \underline{\underline{0}}$$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}) = \underline{\underline{0}}$$

Use the fact that any infinitesimal rotation can be written as a matrix exponential of a skew tensor $\underline{\underline{W}} = -\underline{\underline{W}}^T$

$$\exp(\underline{\underline{A}}) = \sum_{j=0}^{\infty} \frac{1}{j!} \underline{\underline{A}}^j = \underline{\underline{I}} + \underline{\underline{A}} + \frac{1}{2}\underline{\underline{A}}^2 + \ldots$$

We write $\underline{\underline{Q}} = \exp(\epsilon \underline{\underline{W}})$

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}) = \hat{\underline{\underline{\sigma}}}(\exp(\epsilon \underline{\underline{W}})) = \underline{\underline{0}}$$

By definition

$$\mathbb{C}\,\underline{\underline{W}} = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})\,\underline{\underline{W}} = \frac{d}{d\epsilon}\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\underline{\underline{W}})\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\hat{\sigma}\left(\underline{\underline{I}} + \epsilon\underline{\underline{W}} + \frac{1}{2}\epsilon^2\underline{\underline{W}}^2 + \ldots\right)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\hat{\sigma}(\exp(\epsilon\underline{\underline{W}}))\Big|_{\epsilon=0} = \underline{\underline{0}}$$

$$\mathbb{C}\,\underline{\underline{W}} = \underline{\underline{0}}$$

This implies

$$\mathbb{C}\,\underline{\underline{A}} = \mathbb{C}\,\mathrm{sym}(\underline{\underline{A}}) + \cancel{\mathbb{C}\,\mathrm{skew}(\underline{\underline{A}})}$$

$\Rightarrow$ $\mathbb{C}$ has a right minor symmetry $C_{ijkl} = C_{ijlk}$

In summary:

1) Ang. mom. bal.: left minor sym.
2) Frame. indiff: right minor sym.

Elasticity tensor for isotropic solid

Let $\underline{\underline{\hat{S}}}$ be frame indifferent Cauchy stress response for an elastic body with a stress free initial condition. If body is isotropic then $\mathbb{C}$ takes the form

$$\boxed{\mathbb{C}\underline{\underline{H}} = \lambda \operatorname{tr}(\underline{\underline{H}})\underline{\underline{I}} + 2\mu \operatorname{sym}(\underline{\underline{H}})}$$

(Lecture 21)

Linearized Isotropic Elasticity

lin. mom. bal.

$$\rho_0 \ddot{\underline{u}} = \nabla\cdot[\mathbb{C}\nabla\underline{u}] + \rho_0 \underline{b}$$

with $\mathbb{C}\nabla u = \lambda \operatorname{tr}(\nabla\underline{u})\underline{\underline{I}} + 2\mu \operatorname{sym}(\nabla\underline{u})$

need to evaluate $\nabla\cdot\mathbb{C}\nabla\underline{u}$

$$\begin{aligned}
\nabla\cdot[\mathbb{C}\nabla\underline{u}] &= (\lambda u_{k,k}\delta_{ij} + \mu u_{i,j} + \mu u_{j,i})_{,j}\, \underline{e}_i \\
&= (\lambda u_{k,kj}\delta_{ij} + \mu u_{i,jj} + \mu u_{j,ij})\, \underline{e}_i \\
&= \lambda u_{k,ki} + \mu u_{i,jj} + \mu u_{j,ji} \\
&= \lambda \nabla(\nabla\cdot\underline{u}) + \mu\nabla^2\underline{u} + \mu\nabla(\nabla\cdot\underline{u})
\end{aligned}$$

subst. into lin mom. bal.

$$\boxed{\rho_0 \ddot{\underline{u}} = \mu \nabla^2 \underline{u} + (\lambda+\mu)\nabla(\nabla\cdot\underline{u}) + \rho_0 \underline{b}}$$

Navier equation

General linear elastic solid

Stress response function:

$$\hat{\underline{P}}(\underline{F}) = \mathbb{C}\,\underline{\underline{\varepsilon}} \qquad \underline{\underline{\varepsilon}} = \mathrm{sym}(\nabla u)$$

strain-energy density

$$W(\underline{F}) = \tfrac{1}{2}\,\underline{\underline{\varepsilon}} : \mathbb{C}\,\underline{\underline{\varepsilon}}$$

Isotropic model:

$$\mathbb{C}\,\underline{\underline{\varepsilon}} = \lambda\,\mathrm{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{I}} + 2\mu\,\underline{\underline{\varepsilon}}$$

The St. Venant-Kirchhoff model extends this lin. model to large deformation by replacing the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$ with Green Lagrange strain tensor $\underline{E} = \tfrac{1}{2}(\underline{C} - \underline{I})$

Linear: $\hat{\underline{\underline{\sigma}}} = \mathbb{C}\,\underline{\underline{e}} = \lambda\, \mathrm{tr}(\underline{\underline{e}}) + 2\mu\, \underline{\underline{e}}$ $\qquad \underline{\underline{e}} = \mathrm{sym}(\nabla u)$

Non-lin: $\hat{\underline{\underline{\Sigma}}} = y = \lambda\, \mathrm{tr}(\underline{\underline{E}}) + 2\mu\, \underline{\underline{E}}$ $\qquad \underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$

Thank you