Lecture 2: Tensor analysis

Logistics: - Office hours Mon 1:30 - 2 pm

Wed 3:00 - 3:30 pm

- Online for now

Last time: - Vector review $\underline{a} \cdot \underline{b}$ $\underline{a} \times \underline{b}$

- Index notation δ_{ij} ϵ_{ijk}

Today: - Tensors / algebra

- dyadic product / representation
- tensor algebra in index notation
- tensor scalar product
- projection, reflection

Second-order tensors

Linear operator: $\quad \underline{v} = \underline{\underline{A}}\,\underline{u}$

1) $\underline{\underline{A}}(\underline{u}+\underline{v}) = \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v} \qquad$ for all $\underline{u}, \underline{v} \in \mathcal{V}$

2) $\underline{\underline{A}}(\alpha\,\underline{u}) = \alpha\,\underline{\underline{A}}\,\underline{u} \qquad$ for all $\alpha \in \mathbb{R}$, $\underline{u} \in \mathcal{V}$

Example: $\underline{\underline{A}}$ maps every $\underline{v} \in \mathcal{V}$ into $\underline{n} \neq \underline{0} \in \mathcal{V}$

Is $\underline{\underline{A}}$ a tensor?

$$\underline{w} = \underline{u} + \underline{v}$$

$$\underline{\underline{A}}\,\underline{w} \stackrel{?}{=} \underline{\underline{A}}\,\underline{u} + \underline{\underline{A}}\,\underline{v}$$

$$\underline{n} \neq \underline{n} + \underline{n}$$

$\Rightarrow$ $\underline{\underline{A}}$ not tensor

not linear

Tensor algebra

For all $\underline{v} \in \mathcal{V}$ we define

1) $(\alpha\,\underline{\underline{A}})\,\underline{v} = \underline{\underline{A}}\,(\alpha\,\underline{v}) \qquad$ scalar mult.

2) $(\underline{\underline{A}} + \underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,\underline{v} + \underline{\underline{B}}\,\underline{v} \qquad$ tensor sum

3) $(\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{v} = \underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{v}) \qquad$ tensor product

Note: also tensor scalar product $\underline{\underline{A}} : \underline{\underline{B}}$

The set of all second-order tensors $\mathcal{V}^2$ is vector space

1 $\alpha \underline{\underline{A}} \in \mathcal{V}^2$ for all $\alpha \in \mathbb{R}$ $\underline{\underline{A}} \in \mathcal{V}^2$

2) $\underline{\underline{A}} + \underline{\underline{B}} \in \mathcal{V}^2$

3) $\underline{\underline{A}}\,\underline{\underline{B}} \in \mathcal{V}^2$

$\Rightarrow$ all these operations produce another tensor

Q: What is a basis for $\mathcal{V}^2$?

Two tensors $\underline{\underline{A}}, \underline{\underline{B}} \in \mathcal{V}^2$ are equal

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{B}}\,\underline{v} \quad \text{for all } \underline{v} \in \mathcal{V}$$

Zero tensor: $\underline{\underline{0}}\,\underline{v} = \underline{0} \quad \forall \ \underline{v} \in \mathcal{V}$

Identity tensor: $\underline{\underline{I}}\,\underline{v} = \underline{v} \quad \forall \ \underline{v} \in \mathcal{V}$

Representation of tensor

In frame $\{\underline{e}_i\} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\boxed{S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j}$$

Matrix representation of $\underline{\underline{S}}$ in $\{\underline{e}_i\}$

$$[\underline{\underline{S}}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

Note $\quad [\underline{\underline{S}}]_{ij} = S_{ij}$

Consider: $\quad \underline{v} = \underline{\underline{S}}\, \underline{u} \qquad \underline{v} = v_k \underline{e}_k, \; \underline{u} = u_j \underline{e}_j$

$$v_k \underline{e}_k = \underline{\underline{S}}\,(u_j \underline{e}_j) = \underline{\underline{S}}\, \underline{e}_j \; u_j$$

$$v_k \underbrace{\underline{e}_i \cdot \underline{e}_k} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j \, u_j$$

$$v_k \; \delta_{ik} = v_i = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j \; u_j$$

$$\boxed{v_i = S_{ij} \, u_j} \qquad \checkmark$$

Dyadic Products

The dyadic product of $\underline{a}, \underline{b} \in \mathcal{V}$ is $\underline{a} \otimes \underline{b} \in \mathcal{V}^2$ defined by

$$\boxed{(\underline{a} \otimes \underline{b})\,\underline{v} = \underbrace{(\underline{b} \cdot \underline{v})}_{\alpha}\,\underline{a}} \qquad \forall\ \underline{v} \in \mathcal{V}$$

$$\underline{a} \otimes \underline{b} = \underline{a}\,\underline{b}$$

Has the form $\quad \underline{\underline{A}}\,\underline{v} = \alpha\,\underline{a}$

$$A_{ij} v_j = \alpha\, a_i$$

$$\alpha = \underline{b} \cdot \underline{v} = b_j v_j$$

$$A_{ij} = [\underline{a} \otimes \underline{b}]_{ij}$$

$$[\underline{a} \otimes \underline{b}]_{ij}\, v_j = b_j v_j a_i$$

$$= (a_i b_j)\, v_j$$

$$\Rightarrow \boxed{[\underline{a} \otimes \underline{b}]_{ij} = a_i b_j}$$

$$[\underline{a} \otimes \underline{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \underline{a}\,\underline{b}^T$$

Note: $\underline{a}^T \cdot \underbrace{\underline{\underline{S}}\,\underline{b}}_{\underline{v}}$ in tensor notation not explicit if vectors are row or column

instead use $\otimes$, $\cdot$

Linearity of dyadic product:

$$(\underline{a} \otimes \underline{b})(\alpha \underline{u} + \beta \underline{v}) = \alpha (\underline{a} \otimes \underline{b}) \underline{u} + \beta (\underline{a} \otimes \underline{b}) \underline{v}$$

Product of two dyadics:

$$\boxed{(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{c} \cdot \underline{b})(\underline{a} \otimes \underline{d})}$$

Basis for $\mathcal{V}^2$

Given any frame $\{\underline{e}_i\}$ the 9 dyadics $\{\underline{e}_i \otimes \underline{e}_j\}$ form a basis for $\mathcal{V}^2$.

Any $\underline{\underline{S}} \in \mathcal{V}^2$ can be written as linear combination

$$\boxed{\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j} \qquad S_{ij} = \underline{e}_i \cdot \underline{\underline{S}}\, \underline{e}_j$$

Check $\underline{v} = \underline{\underline{S}}\,\underline{u}$ $\underline{v} = v_i\,\underline{e}_i$ $\underline{u} = u_k\,\underline{e}_k$

$$v_i\,\underline{e}_i = S_{ij}(\underline{e}_i \otimes \underline{e}_j)\,(u_k\,\underline{e}_k)$$
$$= S_{ij}\,u_k\,(\underline{e}_i \otimes \underline{e}_j)\,\underline{e}_k$$
$$= S_{ij}\,u_k\,\delta_{jk}\,\underline{e}_i$$
$$v_i\,\underline{e}_i = S_{ij}\,u_j\,\underline{e}_i \quad \checkmark$$

Tensor algebra in components

Addition: $\underline{\underline{H}} = \underline{\underline{S}} + \underline{\underline{T}}$

$$H_{ij}(\underline{e}_i \otimes \underline{e}_j) = S_{ij}(\underline{e}_i \otimes \underline{e}_j) + T_{ij}(\underline{e}_i \otimes \underline{e}_j)$$
$$= (S_{ij} + T_{ij})(\underline{e}_i \otimes \underline{e}_j)$$
$$\Rightarrow \boxed{H_{ij} = S_{ij} + T_{ij}}$$

Scalar multiplication: $\underline{\underline{H}} = \alpha\,\underline{\underline{S}}$ $H_{ij} = \alpha\,S_{ij}$

Product: $\underline{\underline{H}} = \underline{\underline{S}}\,\underline{\underline{T}}$

$$\underline{\underline{H}} = S_{ij}(\underline{e}_i \otimes \underline{e}_j)\, T_{kl}(\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl} (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl} \underbrace{(\underline{e}_j \cdot \underline{e}_k)}\, (\underline{e}_i \otimes \underline{e}_l)$$

$$= S_{ij} T_{kl} \quad \delta_{jk} \; (\underline{e}_i \otimes \underline{e}_l)$$

$$= S_{ij} T_{jl} (\underline{e}_i \otimes \underline{e}_l)$$

$$H_{il}(\underline{e}_i \otimes \underline{e}_l) = S_{ij} T_{jl} (\underline{e}_i \otimes \underline{e}_l)$$

$$\boxed{H_{il} = S_{ij} T_{jl}}$$

Transpose of Tensor

$$\boxed{\underline{\underline{S}}\,\underline{u} \cdot \underline{v} = \underline{u} \cdot \underline{\underline{S}}^T \underline{v}} \qquad \forall \; \underline{u}, \underline{v} \in V$$

implies $\quad S^T_{ij} = S_{ji}$

for proof see notes

Properties: $\quad (\underline{\underline{A}}^T)^T = \underline{\underline{A}}$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$$

$$(\underline{u} \otimes \underline{v})^T = (\underline{v} \otimes \underline{u})$$

$\underline{\underline{S}}$ is symmetric if $\underline{\underline{S}} = \underline{\underline{S}}^T$ $\quad S_{ij} = S_{ji}$

$\underline{\underline{S}}$ is skew symmetric if $\underline{\underline{S}} = -\underline{\underline{S}}^T$ $\quad S_{ij} = -S_{ji}$

Sym. Skew decomposition

$$\boxed{\begin{aligned} \underline{\underline{S}} &= \underline{\underline{E}} + \underline{\underline{W}} \\ \underline{\underline{E}} &= \tfrac{1}{2}\left(\underline{\underline{S}} + \underline{\underline{S}}^T\right) \\ \underline{\underline{W}} &= \tfrac{1}{2}\left(\underline{\underline{S}} - \underline{\underline{S}}^T\right) \end{aligned}}$$

$$\underline{\underline{E}} = \underline{\underline{E}}^T$$

$$\underline{\underline{W}} = -\underline{\underline{W}}^T$$

Note: $\underline{\underline{W}} = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix}$ only 3 indep. comp.

$$\underline{\underline{W}}\,\underline{v} = \underline{w} \times \underline{v} \qquad \forall\, \underline{v} \in \mathcal{V}$$

$\underline{w}$ is axial vector of $\underline{\underline{W}}$

$$W_{ij} = -\epsilon_{ijk} w_k$$

$$w_k = -\tfrac{1}{2}\epsilon_{ijk} W_{ij}$$

Trace of tensor

$$\mathrm{tr}(\underline{a} \otimes \underline{b}) = \underline{a} \cdot \underline{b} = a_i b_i$$

this implies $\underline{\underline{A}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j$

$$\boxed{\mathrm{tr}(\underline{\underline{A}}) = A_{ii} = A_{11} + A_{22} + A_{33}}$$

$$\mathrm{tr}(A_{ij}\, \underline{e}_i \otimes \underline{e}_j) = A_{ij} \underbrace{\mathrm{tr}(\underline{e}_i \otimes \underline{e}_j)}_{e_i \cdot e_j = \delta_{ij}} = A_{ij}\, \delta_{ij} = A_{ii}$$

Properties: $\mathrm{tr}(\underline{\underline{A}}^T) = \mathrm{tr}(\underline{\underline{A}})$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}} + \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}) + \mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\alpha \underline{\underline{A}}) = \alpha\, \mathrm{tr}(\underline{\underline{A}})$$

Decomposition: $\underline{\underline{A}} = \alpha \underline{\underline{I}} + \mathrm{dev}\, \underline{\underline{A}}$

Spherical tensor : $\alpha \underline{\underline{I}} = \frac{1}{3} \mathrm{tr}(A)\, \underline{\underline{I}}$

Deviatoric tensor: $\mathrm{dev}\, \underline{\underline{A}} = \underline{\underline{A}} - \alpha \underline{\underline{I}}$

$$\mathrm{tr}(\mathrm{dev}\, \underline{\underline{A}}) = 0$$

Tensor scalar product / Contraction

$$\boxed{\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}}) = A_{ij} B_{ij}}$$ scalar !

The index expression follows as

$$\underline{\underline{A}} : \underline{\underline{B}} = tr(\underline{\underline{A}}^T \underline{\underline{B}})$$

$$\underline{\underline{A}}^T \underline{\underline{B}} = A_{ji} (\underline{e}_i \otimes \underline{e}_j) \; B_{kl} (\underline{e}_k \otimes \underline{e}_l)$$

$$= A_{ji} B_{kl} \; (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l)$$

$$= A_{ji} B_{kl} \; \underbrace{(\underline{e}_j \cdot \underline{e}_k)}\; (\underline{e}_i \otimes \underline{e}_l)$$

$$= A_{ji} B_{kl} \quad \delta_{jk} \; (\underline{e}_i \otimes \underline{e}_l)$$

$$= A_{ji} B_{jl} \; (\underline{e}_i \otimes \underline{e}_l)$$

$$tr(A_{ji} B_{jl} \; \underline{e}_i \otimes \underline{e}_l) = A_{ji} B_{ji} \quad \checkmark$$

$$tr(\underline{\underline{S}}) = S_{ii} \qquad\qquad i = l$$

Properties: 1) $\underline{\underline{A}} : \underline{\underline{B}} = \underline{\underline{B}} : \underline{\underline{A}}$

2) $(\underline{a} \otimes \underline{b}) : (\underline{c} \otimes \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d})$

Common norm for tensor

$$|\underline{\underline{A}}| = \sqrt{\underline{\underline{A}} : \underline{\underline{A}}} \geq 0$$

End of class

Determinant & Inverse

$$\det(\underline{\underline{A}}) = \det \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} [\underline{\underline{A}}]_{i1} [\underline{\underline{A}}]_{j2} [\underline{\underline{A}}]_{k3}$$

Properties: $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}})$

$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$

$\det(\alpha \underline{\underline{A}}) = \alpha^n \det(\underline{\underline{A}})$ $\quad \underline{\underline{A}}$ $n \times n$

$\underline{\underline{A}}$ is singular if $\det \underline{\underline{A}} = 0$

If A is invertible ($\det \underline{\underline{A}} \neq 0$)

$$\boxed{\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}}$$

Properties: $(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1} \underline{\underline{A}}^{-1}$

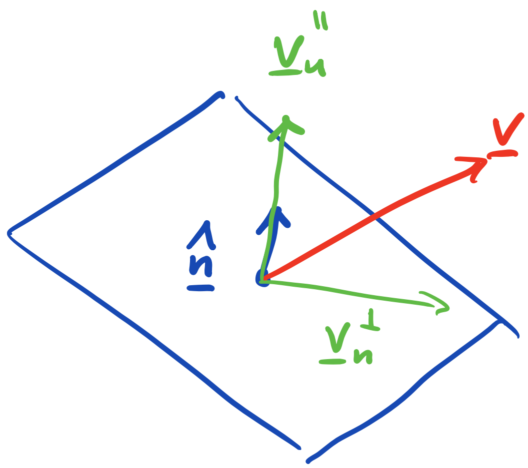
$$(\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$$

$$(\underline{\underline{A}}^{-1})^{T} = (\underline{\underline{A}}^{T})^{-1} = \underline{\underline{A}}^{-T}$$

$$(\alpha \underline{\underline{A}})^{-1} = \frac{1}{\alpha}\, \underline{\underline{A}}^{-1}$$

$$\det(\underline{\underline{A}}^{-1}) = \frac{1}{\det(\underline{\underline{A}})}$$

Projection tensors



$$\underline{v} = \underline{v}_n^{\parallel} + \underline{v}_n^{\perp}$$

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \underline{\hat{n}})\, \hat{n}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel}$$

Projection tensors using dyadic property

$$\underline{v}_n^{\parallel} = (\underline{v} \cdot \underline{\hat{n}})\, \underline{\hat{n}} = (\underline{\hat{n}} \otimes \underline{\hat{n}})\, \underline{v} = \underline{\underline{P}}_n^{\parallel}\, \underline{v}$$

$$\underline{v}_n^{\perp} = \underline{v} - \underline{v}_n^{\parallel} = \underline{\underline{I}}\,\underline{v} - \underline{\underline{P}}_n^{\parallel}\,\underline{v} = (\underline{\underline{I}} - \underline{\hat{n}} \otimes \underline{\hat{n}})\, \underline{v} = \underline{\underline{P}}_n^{\perp}\, \underline{v}$$

$$\boxed{\underline{\underline{P}}_n^{\parallel} = \underline{\hat{n}} \otimes \underline{\hat{n}}} \qquad \boxed{\underline{\underline{P}}_n^{\perp} = \underline{\underline{I}} - \underline{\hat{n}} \otimes \underline{\hat{n}}}$$

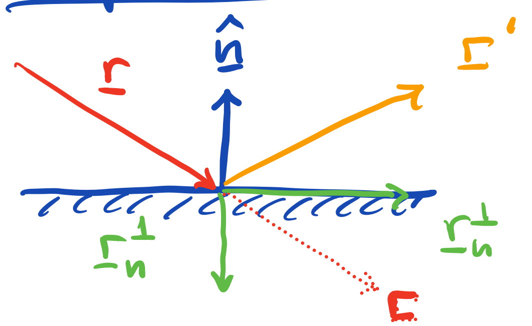
Properties: $\underline{\underline{P}} = \underline{\underline{P}}^T$ symmetric

$$\underline{\underline{P}}^2 = \underline{\underline{P}}$$

$$\underline{\underline{P}}_n^{\parallel} + \underline{\underline{P}}_n^{\perp} = \underline{\underline{I}}$$

$$\underline{\underline{P}}_n^{\parallel}\, \underline{\underline{P}}_n^{\perp} = \underline{\underline{0}}$$

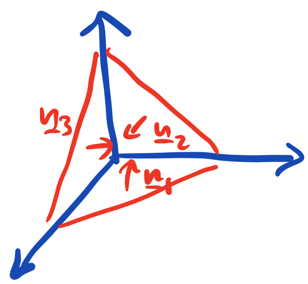
Reflection tensor



$$\underline{r}' = \underline{r}_n^{\perp} - \underline{r}_n^{\parallel}$$

$$= \underbrace{\left(\underline{\underline{I}} - 2\, \underline{\hat{n}} \otimes \underline{\hat{n}}\right)}_{\underline{\underline{R}}_n} \underline{r}$$

Corner reflector



Inverts direction of any ray that reflects of all 3 surfaces.

$$\underline{r}''' = \underline{\underline{R}}_1 \underline{\underline{R}}_2 \underline{\underline{R}}_3 \underline{r}$$

$$\underline{r}''' = -\underline{r}$$