Lecture 3: Tensor algebra

Logistics: - HW1 due 7/7 ✓

- HW2 will be posted

Last time: - Tensor algebra

- Dyadic products $\underline{a} \otimes \underline{b}$
- Transpose / Sym.-Skew decomp.
- Trace / Scalar product.
- Determinant / Inverse
- Projection tensors

Today: - Orthogonal tensors

- Change in basis / representation
- Invariance of trace & determinant
- Eigenproblem / Spectral decomp.
- Tensor square root / Polar decomp.

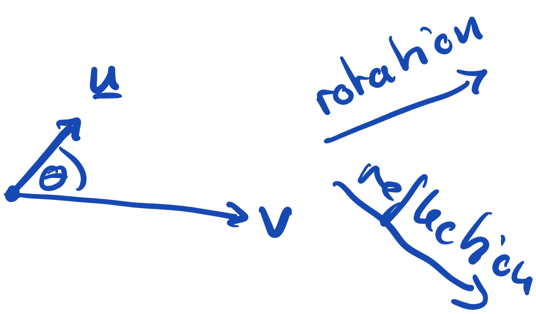
Orthogonal tensors

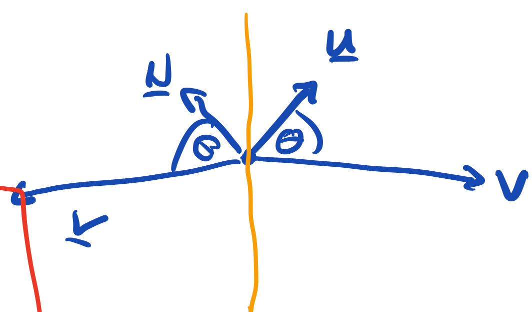
orth. tensor $\underline{\underline{Q}} \in \mathcal{V}^2$ it a linear transf.

$$\underline{u} \cdot \underline{v} = (\underline{\underline{Q}}\,\underline{u}) \cdot (\underline{\underline{Q}}\,\underline{v}) \quad \text{for all } \underline{u}, \underline{v} \in \mathcal{V}$$

$\underline{u} \cdot \underline{v} = |\underline{u}|\,|\underline{v}| \cos(\theta)$

$\Rightarrow$ preserves lengths and angle





Properties:

$$\underline{\underline{Q}}^T = \underline{\underline{Q}}^{-1}$$

$$\underline{\underline{Q}}^T \underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}}$$

$$\det(\underline{\underline{Q}}) = \pm 1$$

$\det(\underline{\underline{I}}) = 1 \Rightarrow \det(\underline{\underline{Q}}^T \underline{\underline{Q}}) = \det(\underline{\underline{Q}}^T)\det(\underline{\underline{Q}}) =$

$= \det(Q)^2 = 1$

if $\det(\underline{\underline{Q}}) = 1 \rightarrow$ rotation (pres. handedness)

$\det(\underline{\underline{Q}}) = -1 \rightarrow$ reflection (changes. handedness)

Change in basis

Both $\underline{v} \in \mathcal{V}$ and $\underline{\underline{T}} \in \mathcal{V}^2$ are invariant upon change of basis, but their representation $[\underline{v}]$ and $[\underline{\underline{S}}]$ are not.

Consider $\{\underline{e}_i\}$ and $\{\underline{e}_i'\}$

representation of $\underline{e}_j'$ in $\{\underline{e}_i\}$

$$\underline{e}_j' = (\underline{e}_1 \cdot \underline{e}_j')\,\underline{e}_1 + (\underline{e}_2 \cdot \underline{e}_j')\,\underline{e}_2 + (\underline{e}_3 \cdot \underline{e}_j')\,\underline{e}_3$$
$$= (\underline{e}_i \cdot \underline{e}_j')\,\underline{e}_i$$

$$\underline{e}_j' = A_{ij}\,\underline{e}_i \qquad \boxed{A_{ij} = \underline{e}_i \cdot \underline{e}_j'}$$

↑ note transpose $\qquad \underline{\underline{A}}\,\underline{v} = A_{ij} v_j$

Here $\underline{\underline{A}}$ is change of basis tensor

$$\boxed{\underline{\underline{A}} = A_{ij}\,\underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}_j'}$$

Note: $\underline{e}_i \cdot \underline{e}_j = \delta_{ij} \quad \underline{e}_i \cdot \underline{e}_j' \neq \delta_{ij}$

$$[\underline{\underline{A}}]_{ij} = A_{ij} \neq [A]'_{ij}$$

Similarly we can express $\underline{e}_i$ in $\{\underline{e}'_k\}$

$$\underline{e}_i = (\underline{e}_i \cdot \underline{e}'_k)\, \underline{e}'_k = A_{ik}\, \underline{e}'_k$$

(↑ not transposed)

We have: $\underline{e}'_j = A_{ij}\, \underline{e}_i \qquad \underline{e}_i = A_{ik}\, \underline{e}'_k$

$\underline{e}'_j = \underbrace{A_{ij} A_{ik}}\, \underline{e}'_k$	$\underline{e}_i = \underbrace{A_{ik} A_{lk}}\, \underline{e}_l$
$\underline{e}'_j = \delta_{jk}\, \underline{e}'_k$	$\underline{e}_i = \delta_{il}\, \underline{e}_l$
$\Rightarrow A_{ij} A_{ik} = \delta_{jk}$	$\Rightarrow A_{ik} A_{lk} = \delta_{il}$

$\underline{\underline{A}}\, \underline{\underline{B}} \Rightarrow A_{ij} B_{jk}$

$$\boxed{\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}\, \underline{\underline{A}}^T = \underline{\underline{I}}}$$

$\underline{\underline{A}}$ is orthogonal

since both $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ are right-handed

$\Rightarrow \underline{\underline{A}}$ is rotation $\Rightarrow \det \underline{\underline{A}} = 1$

Change in representation

Consider $\underline{v} \in \mathcal{V}$ and $\underline{\underline{S}} \in \mathcal{V}^2$

with representations

$[\underline{v}] \quad [\underline{\underline{S}}] \quad$ in $\{\underline{e}_i\}$

$[\underline{v}]' \quad [\underline{\underline{S}}]' \quad$ in $\{\underline{e}'_i\}$

$[\underline{v}] \neq [\underline{v}]' \qquad [\underline{\underline{S}}] \neq [\underline{\underline{S}}]'$

Then $[\underline{v}] = [\underline{\underline{A}}][\underline{v}]' \qquad [\underline{v}]' = [\underline{\underline{A}}]^T[\underline{v}]$

to see this $\underline{v} = v_i \underline{e}_i = v'_j \underline{e}'_j$

with $\underline{e}'_j = A_{ij} \underline{e}_i$ substitute

$$v_i \underline{e}_i = v'_j A_{ij} \underline{e}_i$$

$$v_i = A_{ij} v'_j$$

Similarly

$$[\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T$$

$$[\underline{\underline{S}}]' = [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}]$$

$\Rightarrow$ HW2

Show $tr(\underline{\underline{S}})$ and $det(\underline{\underline{S}})$ are invariant under change in basis.

Invariance of trace

For $\underline{\underline{S}} \in \mathcal{V}^2$ with $[\underline{\underline{S}}]$ in $\{\underline{e}_i\}$ and $[\underline{\underline{S}}]'$ in $\{\underline{e}_i'\}$

$$\boxed{tr[\underline{\underline{S}}] = tr[S]'}$$

$$tr[S] = [S]_{ii}$$

$$[\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T$$

$$[S]_{ij} = [\underline{\underline{A}}]_{ik}[\underline{\underline{S}}]'_{kl}[A]_{jl} \qquad [\underline{\underline{A}}]^T_{lj} = [\underline{\underline{A}}]_{jl}$$

$$tr[\underline{\underline{S}}] = [S]_{ii} = [A]_{ik}[\underline{\underline{S}}]'_{kl}[\underline{\underline{A}}]_{il}$$
$$= \underbrace{[\underline{\underline{A}}]_{ik}[\underline{\underline{A}}_{il}]}_{\delta_{kl}}[\underline{\underline{S}}]'_{kl} = [S]'_{kk} = tr[\underline{\underline{S}}]'$$

Invariance of determinant

$$\det[\underline{\underline{S}}] = \det[\underline{\underline{S}}]' \quad \Rightarrow \text{HW2}$$

$$\uparrow$$
$$A\,S\,A^T$$

$\Rightarrow$ constitutive theory

Eigenvalues & eigenvectors of tensors

$(\lambda, \underline{v})$ eigenpair of $\underline{\underline{S}} \in \mathcal{V}^2$

$$\underline{\underline{S}}\,\underline{v} = \lambda\,\underline{v}$$

λ = eigenvalue $\quad \underline{v}$ = eigenvector

λ's are roots of char. polynomial

$$p(\lambda) = \det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For each λ_p we have one or more $\underline{v}_p$

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = 0$$

Continuum mechanics interested in symmetric tensors $\underline{\underline{S}} = \underline{\underline{S}}^T$

For $\underline{\underline{S}} = \underline{\underline{S}}^T$:

1) All λ's real
2) All λ's are pos. ($\underline{\underline{S}}$ is sym. pos. def)
3) All $\underline{v}_p$ corresponding to distinct λ_p's are orthonormal

$\underline{\underline{S}}$ is sym. pos. def. (s.p.d)

if $\underline{v} \cdot \underline{\underline{S}}\,\underline{v} > 0$ for all $\underline{v} \in \mathbb{R}^3$

use def of eigen pair $\underline{\underline{S}}\,\underline{v} = \lambda \underline{v}$

$$\underline{v} \cdot (\lambda \underline{v}) > 0$$

$$\lambda \underbrace{|\underline{v}|^2}_{>0} > 0 \quad \Rightarrow \quad \lambda > 0$$

Consider $(\lambda, \underline{v})$ and $(\omega, \underline{u})$

$\lambda \neq \omega \qquad \underline{\underline{S}}\,\underline{v} = \lambda \underline{v} \qquad \underline{\underline{S}}\,\underline{u} = \omega \underline{u}$

Consider $\lambda(\underline{v} \cdot \underline{u}) = (\lambda \underline{v} \cdot \underline{u}) = (\underline{\underline{S}}\,\underline{v} \cdot \underline{u})$

$$\underline{v} \cdot \underline{\underline{S}}\,\underline{u} = \underline{v} \cdot (\omega \underline{u}) = \omega (\underline{v} \cdot \underline{u})$$

$$\lambda(\underline{v} \cdot \underline{u}) = \omega(\underline{v} \cdot \underline{u}) \qquad \lambda \neq \omega$$

$$\Rightarrow \underline{v} \cdot \underline{u} = 0$$

Spectral decomposition

If $\underline{\underline{S}} \in \mathcal{V}$ is $\underline{\underline{S}} = \underline{\underline{S}}^T$ there exist a frame $\{\underline{v}_i\}$ such that

$$\boxed{\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i}$$

$$[\underline{\underline{S}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Since $\underline{v}_i$ are orthonormal: $\underline{\underline{I}} = \underline{v}_i \otimes \underline{v}_i$

$$\underline{\underline{S}} = \underline{\underline{S}}\,\underline{\underline{I}} = \underline{\underline{S}}(\underline{v}_i \otimes \underline{v}_i) = (\underline{\underline{S}}\underline{v}_i \otimes \underline{v}_i)$$

$$= \sum_{i=1}^{3} (\lambda_i v_i) \otimes \underline{v}_i$$

$$= \sum_{i=1}^{3} \lambda_i (\underline{v}_i \otimes \underline{v}_i)$$

used $\quad \underline{\underline{A}}(\underline{u} \otimes \underline{v}) = (\underline{\underline{A}}\underline{u} \otimes \underline{v}) \Rightarrow$ HW2

$\quad \alpha(\underline{u} \otimes \underline{v}) = (\alpha \underline{u} \otimes \underline{v})$

The principal invariants of $\underline{\underline{S}} \in \mathcal{V}^2$ are

$$I_1(\underline{\underline{S}}) = tr(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3$$
$$I_2(\underline{\underline{S}}) = \frac{1}{2}\left((tr\,\underline{\underline{S}})^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$
$$I_3(\underline{\underline{S}}) = det(\underline{\underline{S}}) = \lambda_1 \lambda_2 \lambda_3$$

These three scalars are frame invariant

Set of invariants $I_S = \{I_1(\underline{\underline{S}}), I_2(\underline{\underline{S}}), I_3(\underline{\underline{S}})\}$

Rewrite characteristic polynomial

$$det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = -\lambda^3 + I_1(\underline{\underline{S}})\lambda^2 - I_2(\underline{\underline{S}})\lambda + I_3(\underline{\underline{S}}) = 0$$

write out char. poly and collect terms

Consider: $\underline{\underline{S}}\,\underline{\underline{S}}\,\underline{v} = \lambda\,\underline{\underline{S}}\,\underline{v} = \lambda^2 \underline{v}$

in general $\underline{\underline{S}}^\alpha \underline{v} = \lambda^\alpha \underline{v}$

multiply char. poly by $\underline{v}$

$$-\lambda^3 \underline{v} + I_1 \lambda^2 \underline{v} - I_2 \lambda v + I_3 \underline{v} = 0\,\underline{v} = 0$$

$$-\underline{\underline{S}}^3\underline{v} + I_1 \underline{\underline{S}}^2 \underline{v} - I_2 \underline{\underline{S}} + I_3 \underline{v} = 0 \quad \text{for all } \underline{v} \in \mathcal{V}$$

$$\Rightarrow \boxed{-\underline{\underline{S}}^3 + I_1(\underline{\underline{S}})\underline{\underline{S}}^2 - I_2(\underline{\underline{S}})\underline{\underline{S}} + I_3(\underline{\underline{S}}) = 0}$$

Cayley-Hamilton theorem

Tensor square root

If $\underline{\underline{C}} \in \mathcal{V}^2$ is s.p.d with eigen pair $(\lambda, \underline{v})$ then there is a unique tensor $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$ defined as

$$\boxed{\underline{\underline{U}} = \sqrt{\underline{\underline{C}}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, \underline{v}_i \otimes \underline{v}_i}$$

Polar decomposition

Any tensor $\underline{\underline{F}} \in \mathcal{V}^2$ with $\det(\underline{\underline{F}}) > 0$ has a right & left polar decomp.

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$$

where $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d.

and $\underline{\underline{R}}$ is a rotation.

To see this consider

$\det(\underline{\underline{F}}) > 0 \Rightarrow \underline{\underline{F}}\,\underline{v} \neq 0 \quad \underline{v} \in \mathcal{V}^2$

$\det(\underline{\underline{F}}^T) > 0 \Rightarrow \underline{\underline{F}}^T \underline{v} \neq 0 \quad$ "

$(\underline{\underline{F}}\,\underline{v}) \cdot (\underline{\underline{F}}\,\underline{v}) > 0$

$(\underline{\underline{F}}\,\underline{v})^T (\underline{\underline{F}}\,\underline{v}) = \underline{v}^T \underline{\underline{F}}^T \underline{\underline{F}}\,\underline{v} > 0$

$\underline{v} \cdot \underline{\underline{U}}^2 \underline{v} > 0$

Show $\underline{\underline{R}}$ is rotation

$\underline{\underline{F}}\,\underline{\underline{U}}^{-1} = \underline{\underline{R}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1} \Rightarrow \underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$ ($\underline{\underline{U}}\,\underline{\underline{U}}^{-1} = I$)

$\det(\underline{\underline{R}}) = \det(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \dfrac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$

Show $\underline{\underline{R}}$ is orthonormal

$\underline{\underline{R}}^T \underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T (\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) =$

$= \underline{\underline{U}}^{-T} \underline{\underline{F}}^T \underline{\underline{F}}\,\underline{\underline{U}}^{-1} \qquad U = \text{sym.}$

$= \underline{\underline{U}}^{-1} \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}\,\underline{\underline{U}}^{-1} = \underline{\underline{U}}^{-1} U\, U\, U^{-1} = \underline{\underline{I}} \cdot \underline{\underline{I}} = \underline{\underline{I}}$

u^2