Lecture 4: Tensor calculus

7/11/21

Logistics: - HW 1 graded

- HW 2 due Thursday

Last time: - orthogonal tensors $Q^T = Q^{-1}$

- change in basis $[v] = [\underline{A}][\underline{v}]'$
- invariance of trace & determinant
- Eigenproblem & spectral decomp.
- Tensor square root / Polar decomp.

Today: - Differentiation of tensor fields

div, grad, curl and all that

"Field" is a object that is function of space

scalar fields: $\phi(\underline{x})$ Temp, density

vector fields: $\underline{v}(\underline{x})$ velocity, displacement

tensor fields: $\underline{\underline{S}}(\underline{x})$ stress, conductivity

$\Rightarrow$ today review & extension of vector calc

Gradients

Gradient of scalar field

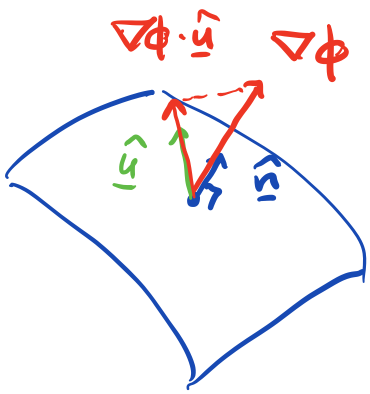
Scalar field $\phi(\underline{x}) \in \mathbb{R}$ is differentiable at $\underline{x}$ if there exists a vector field $\nabla\phi(\underline{x}) \in \mathcal{V}$ such that

$$\phi(\underline{x}+\underline{h}) = \phi(\underline{x}) + \nabla\phi(\underline{x})\cdot\underline{h} + o(|\underline{h}|)$$

by Taylor expansion. $\underline{h} = \epsilon\,\hat{\underline{u}}$ we can write

$$\boxed{\nabla\phi(\underline{x})\cdot\hat{\underline{u}} = \frac{d}{d\epsilon}\,\phi(x+\epsilon\hat{\underline{u}})\Big|_{\epsilon=0}} \quad \forall\ \underline{u}\in\mathcal{V}$$

The vector $\nabla\phi$ is called gradient of ϕ



Consider $\phi(x) = \phi_0$

$\nabla\phi \parallel \hat{\underline{n}}$ in dir. of increasing ϕ.

$$\hat{\underline{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Directional derivative (Gâteaux operator):

$$\boxed{D_{\underline{\hat{u}}}\phi(x) = \frac{d}{d\varepsilon}\,\phi(\underline{x}+\epsilon\,\underline{\hat{u}})\Big|_{\epsilon=0} = \nabla\phi(\underline{x})\cdot\underline{\hat{u}}}$$

Representation in frame $\{\underline{e}_i\}$

$$\phi(\underline{\bar{x}}+\epsilon\,\underline{\hat{u}}) = \phi(\underbrace{\bar{x}_1+\epsilon\,\hat{u}_1}_{x_1},\ \underbrace{\bar{x}_2+\epsilon\,\hat{u}_2}_{x_2},\ \underbrace{\bar{x}_3+\epsilon\,\hat{u}_3}_{x_3})$$

$$\nabla\phi\cdot\underline{\hat{u}} = \frac{d}{d\epsilon}\,\phi(\bar{x}_1+\epsilon\hat{u}_1,\ \bar{x}_2+\epsilon\hat{u}_2,\ \bar{x}_3+\epsilon\hat{u}_3)\Big|_{\epsilon=0}$$
$$= \frac{\partial\phi}{\partial x_1}\frac{dx_1}{d\epsilon} + \frac{\partial\phi}{\partial x_2}\frac{dx_2}{d\epsilon} + \frac{\partial\phi}{\partial x_3}\frac{dx_3}{d\epsilon}\Big|_{\epsilon=0}$$
$$= \frac{\partial\phi}{\partial x_1}u_1 + \frac{\partial\phi}{\partial x_2}u_2 + \frac{\partial\phi}{\partial x_3}u_3$$
$$= \frac{\partial\phi}{\partial x_i}u_i = \phi_{,i}\,u_i = \phi_{,i}\,u_j\,\delta_{ij} = \phi_{,i}\,u_j\,(\underline{e}_i\cdot\underline{e}_j)$$
$$= (\phi_{,i}\,\underline{e}_i)\cdot(u_j\,\underline{e}_j)$$

Gradient in comp: $[\nabla\phi] = \phi_{,i}\,\underline{e}_i = \begin{bmatrix}\frac{\partial\phi}{\partial x_1}\\ \frac{\partial\phi}{\partial x_2}\\ \frac{\partial\phi}{\partial x_3}\end{bmatrix}$

Note: index notation for derivatives

$$\boxed{\frac{\partial\phi}{\partial x_i} = \phi_{,i}}$$ derivative index after comma

Gradient of a vector field

Vector field $\underline{v}(x) \in \mathcal{V}$ is differentiable at $\underline{x}$ if there exists a tensor field $\nabla \underline{v} \in \mathcal{V}^2$ such that $\underline{v}(\underline{x}+\underline{h}) = \underline{v}(\underline{x}) + \nabla \underline{v}(\underline{x})\,\underline{h} + o(|\underline{h}|)$

again by Taylor expansion. So that

$$\boxed{\nabla \underline{v}\, \underline{\hat{u}} = \frac{d}{d\varepsilon}\, \underline{v}(\underline{x} + \epsilon \underline{\hat{u}})\Big|_{\epsilon=0}}$$

In frame $\{\underline{e}_i\}$ the components of $\underline{v}$ are $v_i = v_i(x_1, x_2, x_3)$

$$v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}}) = v_i(\bar{x}_1 + \epsilon \hat{u}_1, \bar{x}_2 + \epsilon \hat{u}_2, \bar{x}_3 + \epsilon \hat{u}_3)$$

use definition

$$\frac{d}{d\epsilon} v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\Big|_{\epsilon=0} = v_{i,1}\hat{u}_1 + v_{i,2}\hat{u}_2 + v_{i,3}\hat{u}_3$$

$$= v_{i,j}\,\hat{u}_j$$

Full vector $\underline{v} = v_i \underline{e}_i$

$$\nabla \underline{v}\, \underline{\hat{u}} = \frac{d}{d\varepsilon} \underline{v}(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\Big|_{\epsilon=0} = \frac{d}{d\varepsilon} v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\, \underline{e}_i\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon} v_i(\overline{x} + \epsilon \hat{\underline{u}})\Big|_{\epsilon=0} \underline{e}_i$$

$$\nabla \underline{v}\, \hat{\underline{u}} = v_{i,j}\, \hat{u}_j\, \underline{e}_i$$

components $\boxed{[\nabla \underline{v}]_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}}$

$$\boxed{\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j}$$

Explicit

$$[\nabla \underline{v}] = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \\ \nabla v_3^T \end{bmatrix}$$

Divergence of a vector field

To any $\underline{v}(x) \in \mathcal{V}$ we associate a scalar field $\nabla \cdot \underline{v} \in \mathbb{R}$ called the divergence

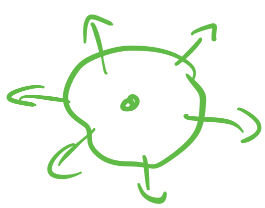
$$\boxed{\nabla \cdot \underline{v} = \mathrm{tr}(\nabla \underline{v})}$$

In frame $\{\underline{e}_i\}$ $\quad \underline{v}(x) = v_i(x)\,\underline{e}_i$, we have

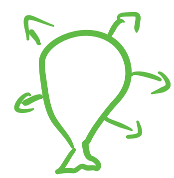
$$\nabla\cdot\underline{v} = tr(\nabla\underline{v}) = tr(v_{i,j}\,\underline{e}_i\otimes\underline{e}_j) = v_{i,i}$$

$$\boxed{\nabla\cdot\underline{v} = v_{i,i}} = v_{1,1} + v_{2,2} + v_{3,3}$$

If $\nabla\cdot\underline{v} = 0$ a field is solenoidal or div. free. Next lecture we'll show that $\nabla\cdot\underline{v}$ is related to (rate) volume change.



$\nabla\cdot\underline{v} > 0$



Divergence of tensor fields

To any $\underline{\underline{S}} \in \mathcal{V}^2$ we associate a vector field $\nabla\cdot\underline{\underline{S}} \in \mathcal{V}^2$ called the divergence of $\underline{\underline{S}}$

$$\boxed{(\nabla\cdot\underline{\underline{S}})\cdot\underline{a} = \nabla\cdot(\underline{\underline{S}}^T\underline{a})} \quad \text{for all const. } \underline{a}\in\mathcal{V}$$

use the def. of vector divergence.

In frame $\{\underline{e}_i\}$ with $\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j$

and $\underline{a} = a_k \underline{e}_k$ we have $\underline{q} = \underline{\underline{S}}\, \underline{a}$

$$q_j = S_{ij} a_i \quad (q_i = S_{ji} a_j)$$

substitute into definition

$$(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a}) = \nabla \cdot \underline{q} = \mathrm{tr}(\nabla \underline{q}) = q_{j,j}$$

$$= S_{ij,j}\, a_i = (S_{ij,j}\, \underline{e}_i) \cdot (a_k \underline{e}_k)$$

by arbitrariness of $\underline{a}$

$$\boxed{\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i}$$

Gradient & Divergence product rules

$\phi \in \mathbb{R} \quad \underline{v} \in \mathcal{V} \quad \underline{\underline{S}} \in \mathcal{V}^2$

$$\boxed{\begin{aligned}
\nabla \cdot (\phi \underline{v}) &= \underline{v} \cdot \nabla \phi + \phi\, \nabla \cdot \underline{v} \\
\nabla \cdot (\phi \underline{\underline{S}}) &= \underline{\underline{S}}\, \nabla \phi + \phi\, \nabla \cdot \underline{\underline{S}} \\
\nabla \cdot (\underline{\underline{S}}^T \underline{v}) &= (\nabla \cdot \underline{\underline{S}}) \cdot \underline{v} + \underline{\underline{S}} : \nabla \underline{v} \\
\nabla (\phi \underline{v}) &= \underline{v} \otimes \nabla \phi + \phi\, \nabla \underline{v}
\end{aligned}}$$

$$\nabla \underline{v} = v_{i,j}\, e_i \otimes e_j$$

Example: $\nabla(\phi \underline{v}) = (\phi v_i)_{,j} \, \underline{e}_i \otimes \underline{e}_j$

$\underline{a} \otimes \underline{b} = a_i b_j \, \underline{e}_i \otimes \underline{e}_j$

$$= (\phi_{,j} v_i + \phi v_{i,j}) \, \underline{e}_i \otimes \underline{e}_j$$

$$= \underline{v} \otimes \nabla\phi + \phi \nabla \underline{v}$$

Curl of a vector field

To any $\underline{v}(\underline{x}) \in \mathcal{V}$ we associate another vector field $\nabla \times \underline{v}$ defined by

$$\boxed{(\nabla \times \underline{v}) \times \underline{a} = \underbrace{(\nabla \underline{v} - \nabla \underline{v}^T)}_{\underline{\underline{T}}} \, \underline{a}} \quad \text{for all const } \underline{a} \in \mathcal{V}$$

$\nabla \underline{v}^T = (\nabla \underline{v})^T$

$\underline{\underline{T}} = 2 \, \text{skew}(\nabla \underline{v})$ and

$\underline{w} = \nabla \times \underline{v}$ is axial vector of $\underline{\underline{T}}$

In index notation

$$w_j = \tfrac{1}{2} \epsilon_{ijk} T_{ik} = \tfrac{1}{2} \epsilon_{ijk} (v_{i,k} - v_{k,i})$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} - \epsilon_{ijk} v_{k,i}), \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} + \epsilon_{kji} v_{k,i}) \qquad i = k \quad k = i$$

$$= \tfrac{1}{2} (\epsilon_{ijk} v_{i,k} + \epsilon_{ijk} v_{i,k})$$

$w_j = \epsilon_{ijk} v_{i,k}$

$$\boxed{\underline{w} = \nabla \times \underline{v} = \epsilon_{ijk} v_{i,k} \underline{e}_j}$$

$$-\epsilon_{ikj} v_{i,k} \underline{e}_j$$

$$-\epsilon_{ijk} v_{i,j} \underline{e}_k$$

Explicitly:

$$\nabla \times \underline{v} = (v_{3,2} - v_{2,3}) \underline{e}_1 + (v_{1,3} - v_{3,1}) \underline{e}_2 + (v_{2,1} - v_{1,2}) \underline{e}_3$$

$\Rightarrow$ curl of velocity field is related to angular velocity

If $\nabla \times \underline{v} = 0 \Rightarrow \underline{v}(\underline{x})$ is irrotational/conservative

We can show

$\boxed{\nabla \times \nabla \phi = 0}$ and $\boxed{\nabla \cdot (\nabla \times \underline{v}) = 0}$

Laplacian

To any scalar field $\phi \in \mathbb{R}$ we associate another scalar field $\Delta\phi = \nabla^2\phi$

$$\boxed{\Delta\phi = \nabla^2\phi = \nabla \cdot \nabla\phi}$$

In frame $\{\underline{e}_i\}$ with $\nabla\phi = \phi_{,i}\,\underline{e}_i$ we have

$$\nabla \cdot \nabla\phi = \mathrm{tr}(\nabla\nabla\phi) = \mathrm{tr}(\phi_{,ij}\,\underline{e}_i \otimes \underline{e}_j) = \phi_{,ii}$$

$$\boxed{\nabla^2\phi = \phi_{,ii}}$$

Laplacian governs steady heat flow

Vector Laplacian

To any $\underline{v}(x) \in \mathcal{V}$ we associate another vector field $\Delta\underline{v} = \nabla^2\underline{v} \in \mathcal{V}$ defined to be

$$\boxed{\Delta\underline{v} = \nabla^2\underline{v} = \nabla \cdot \nabla\underline{v}}$$

In frame $\{\underline{e}_i\}$ $\underline{v} = v_i\,\underline{e}_i$ $\nabla\underline{v} = v_{i,j}\,\underline{e}_i \otimes \underline{e}_j$

and $\nabla \cdot \underline{\underline{S}} = S_{ij,j}\,\underline{e}_i$ so that

$$\boxed{\Delta\underline{v} = \nabla^2\underline{v} = v_{i,jj}\,\underline{e}_i}$$

governs creeping flows.

One useful identity

$$\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v})$$

if $\nabla \cdot \underline{v} = 0$ and $\nabla \times \underline{v} = 0$

$\Rightarrow$ $\nabla^2 \underline{v} = 0$ $\underline{v}$ is harmonic