Lecture 5: Integral theorems & Tensor functions

Logistics: - HW 2 due

- HW3 will be posted
- sorry about office hours yesterday

Last time: - Differentiation of tensor fields

- $\nabla \phi = \phi_{,i} e_i$ $\quad \nabla \underline{v} = v_{i,j} \underline{e}_i \otimes \underline{e}_j$
- $\nabla \cdot \underline{v} = v_{i,i}$ $\quad \nabla \cdot \underline{\underline{S}} = S_{ij,j} \underline{e}_i$
- $\nabla \times \underline{v} = \epsilon_{ijk} v_{i,k} \underline{e}_j$
- $\nabla^2 \phi = \phi_{,ii}$ $\quad \nabla^2 \underline{v} = v_{i,jj} \underline{e}_i$

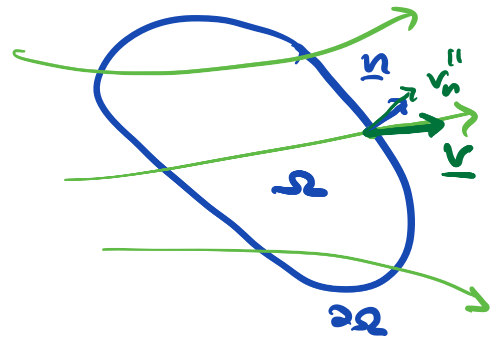
Today: - Integral theorems

- Differentiation of tensor functions

Vector divergence theorem

For any $\underline{v} \in \mathcal{V}$ we have

$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{v}\cdot\hat{\underline{n}}\, dA &= \int_{\Omega} \nabla\cdot\underline{v}\, dV \\ \int_{\partial\Omega} v_i \hat{n}_i\, dA &= \int_{\Omega} v_{i,i}\, dV \end{aligned}}$$

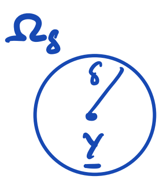


from vector calculus

Physical interpretation

Here $\underline{v}$ as a velocity $\left[\frac{L}{T}\right]$ or as volumetric flux $\left[\frac{L^3}{L^2 T} = \frac{L}{T}\right]$. The units of the lhs $\int \underline{v}\cdot\hat{\underline{n}}\, dA$ are $\left[\frac{L^3}{T}\right]$ $\Rightarrow$ rate at which volume leaving/entering Ω.

Interpretation of $\nabla\cdot v$



$$\int_{\partial\Omega_\delta} \underline{v}\cdot\hat{\underline{n}}\, dA = \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV$$

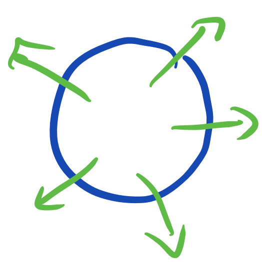
$$\lim_{\delta\to 0} \int_{\Omega_\delta} \nabla\cdot\underline{v}\, dV = V_\delta\, \nabla\cdot\underline{v}\big|_{\underline{y}}$$

$$\boxed{\nabla\cdot\underline{v}\,\Big|_{x} = \lim_{\delta\to 0} \frac{1}{V_\delta}\int_{\partial\Omega} \underline{v}\cdot\hat{\underline{n}}\,dA}$$

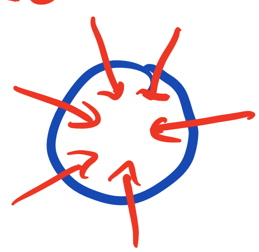
$$\frac{1}{L}\quad \frac{L}{T}\qquad\qquad \underbrace{\frac{1}{L^3}\qquad \frac{L^3}{T}}_{\frac{1}{T}}$$

Divergence is the rate of vol. expansion or contraction per unit volume.

$\nabla\cdot\underline{v} > 0$



$\nabla\cdot v < 0$



Incompressible flows/deformations are solenoidal $\nabla\cdot\underline{v} = 0$

Tensor divergence theorem

For any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ on domain Ω with bnd $\partial\Omega$ we have

$$\boxed{\begin{aligned} \int_{\partial\Omega} \underline{\underline{S}}\,\hat{\underline{n}}\, dA &= \int_{\Omega} \nabla\cdot\underline{\underline{S}}\, dV \\ \int_{\partial\Omega} S_{ij}\hat{n}_j\, dA &= \int_{\Omega} S_{ij,j}\, dV \end{aligned}}$$

To derive this from vector version consider constant vector $\underline{a} \in \mathcal{V}$

$$\underline{a}\cdot\int_{\partial\Omega} \underline{\underline{S}}\,\hat{\underline{n}}\, dA = \int_{\partial\Omega} \underline{a}\cdot\underline{\underline{S}}\hat{\underline{n}}\, dA = \int_{\partial\Omega} (\underline{\underline{S}}^T\underline{a})\cdot\hat{\underline{n}}\, dA$$

where $\underline{\underline{S}}^T\underline{a}$ is vector so we apply div. Thm

$$\int_{\partial\Omega} (\underline{\underline{S}}^T\underline{a})\cdot\hat{\underline{n}}\, dA = \int_{\Omega} \nabla\cdot(\underline{\underline{S}}^T a)\, dV$$

use definition of tensor divergence

$$\int_{\Omega} \nabla\cdot(\underline{\underline{S}}^T\underline{a})\, dV = \int_{\Omega} (\nabla\cdot\underline{\underline{S}})\cdot\underline{a}\, dV$$

$$\int_{\partial\Omega} ((\underline{\underline{S}}^T a)\cdot\hat{\underline{n}})\, dA = \int_{\Omega} (\nabla\cdot\underline{\underline{S}})\cdot\underline{a}\, dV$$

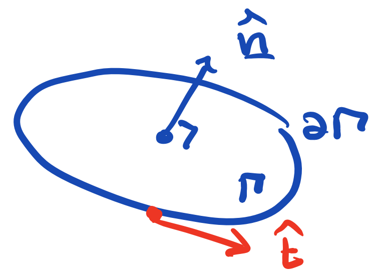
using transpose & that $\underline{a}$ = const

$$\underline{a}\cdot\int_{\partial\Omega}\underline{\underline{S}}\,\hat{\underline{n}}\,dA = \underline{a}\cdot\int_{\Omega}(\nabla\cdot\underline{\underline{S}})\,dV$$

the result follows from arbitraryness of $\underline{a}$

Stokes Thm

Consider surface Γ with boundary $\partial\Gamma$.
unit normal $\hat{\underline{n}}$ and right handed unit tangent $\hat{\underline{t}}$

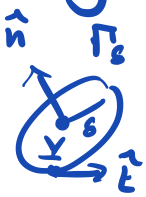


Then for any $\underline{v}(\underline{x}) \in V$ we have

$$\boxed{\int_{\Gamma}(\nabla\times\underline{v})\cdot\hat{\underline{n}}\,dA = \oint_{\partial\Gamma}\underline{v}\cdot\hat{\underline{t}}\,ds}$$

The rhs is called the circulation of $\underline{v}$ around $\partial\Gamma$.

Physical interpretation:



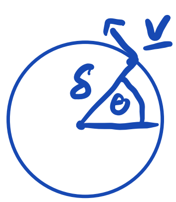
Π_δ is a disk of radius δ around $\underline{y}$. $\hat n \parallel \nabla \times \underline{v}$

$$\int_{\Pi_\delta} (\nabla \times \underline{v}) \cdot \hat n \, dA = \oint_{\partial \Pi} \underline{v} \cdot \hat{\underline{t}} \, ds$$

In limit $\delta \to 0$

$$\overline{\underline{v} \cdot \hat{\underline{t}}}\Big|_y \, 2\pi\delta \simeq \nabla \times \underline{v}\big|_{\underline{y}} \cdot \hat{\underline{n}} \; \pi \delta^2$$

ave. tangential velocity ~ angular velocity



$$\omega = \frac{d\theta}{dt} \qquad |\underline{v}| = \omega\delta$$

$$\Rightarrow \quad \overline{\underline{v} \cdot \hat{t}}\Big|_{\underline{y}} \simeq \omega_y \delta$$

substitute

$$2\cancel{\pi\delta^2}\,\omega \simeq \nabla \times \underline{v}\big|_{y} \cdot \hat n \; \cancel{\pi \delta^2}$$

$$\hat{\underline{n}} = \frac{\nabla \times \underline{v}|_y}{|\nabla \times \underline{v}|_y} \qquad \text{subst.} \quad \frac{(\nabla \times v|_y) \cdot (\nabla \times v|_y)}{1}$$

$$\boxed{2\,\omega_{\underline{y}} = |\nabla \times \underline{v}|_{\underline{y}}}$$

Curl of $\underline{v}$ is twice angular velocity

Derivatives of tensor functions

so far field $\phi(\underline{x})$ $\underline{v}(\underline{x})$ $\underline{\underline{S}}(\underline{x})$

Now we are interested in functions that tensors as input:

- scalar valued tensor functions: $\psi = \psi(\underline{\underline{S}})$
- tensor valued tensor functions: $\underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}(\underline{\underline{S}})$

Derivatives of scalar valued tensor functions.

Typical examples: $\det(\underline{\underline{A}})$ $\operatorname{tr}(A)$

Def: A function $\psi(\underline{A})$ is differentiable at $\underline{\underline{A}}$ if there exists a tensor $D\psi(A)$, s.t by Taylor expansion

$$\psi(\underline{\underline{A}} + \underline{\underline{H}}) = \psi(\underline{\underline{A}}) + D\psi(\underline{\underline{A}}) : \underline{\underline{H}} + o(|H|)$$

or equivalently with $\underline{\underline{H}} = \epsilon \underline{\underline{U}}$

$$\boxed{D\psi(\underline{\underline{A}}) : \underline{\underline{U}} = \frac{d}{d\epsilon} \psi(\underline{\underline{A}} + \epsilon \underline{\underline{U}}) \Big|_{\epsilon = 0}}$$ for all $\underline{\underline{U}} \in \mathcal{V}^2$

$D\psi(\underline{\underline{A}})$ derivative of ψ at $\underline{\underline{A}}$

In a frame $\{\underline{e}_i\}$ we have

$$\boxed{D\psi(\underline{\underline{A}}) = \frac{\partial \psi}{\partial A_{ij}} \underline{e}_i \otimes \underline{e}_j}$$

write $\psi(A_{11}, A_{12}, \dots, A_{33})$

$\underline{\underline{U}} = U_{kl}\, \underline{e}_k \otimes \underline{e}_l$

$$\psi(\bar{\underline{A}} + \epsilon \underline{\underline{U}}) = \psi(\bar{A}_{11} + \epsilon U_{11}, \dots, \bar{A}_{33} + \epsilon U_{33})$$

by chain rule

$$\underline{D\psi(A)} \overset{:\mathcal{U}}{=} \frac{d}{d\varepsilon}\psi(\underbrace{\bar{A}_{11}+\epsilon U_{11}}_{A_{11}}, \ldots, \bar{A}_{33}+\epsilon U_{33})\Big|_{\epsilon=0}$$

$$= \frac{\partial\psi}{\partial A_{11}} U_{11} + \frac{\partial\psi}{\partial A_{12}} U_{12} + \ldots + \frac{\partial\psi}{\partial A_{33}} U_{33}$$

$$= \frac{\partial\psi}{\partial A_{ij}} U_{ij} = \left(\underline{\frac{\partial\psi}{\partial A_{ij}} e_i \otimes e_j}\right) : \left(U_{kl}\, \underline{e}_k \otimes \underline{e}_l\right)$$

Result follows from arbitrariness of $\underline{\underline{U}}$.

Derivative of trace

$\psi(\underline{\underline{A}}) = tr(\underline{\underline{A}}) = \underline{A_{ii}}$ $\qquad D\psi(A) = \frac{\partial\psi}{\partial A_{ij}}$

$$D\,tr(A) = \frac{\partial A_{ii}}{\partial A_{kl}} \underline{e}_k \otimes \underline{e}_l = \delta_{ik}\,\delta_{il}\, \underline{e}_k \otimes \underline{e}_l$$

$$= \underline{e}_i \otimes \underline{e}_j = \underline{\underline{I}}$$

$$\boxed{D\,tr(\underline{\underline{A}}) = \underline{\underline{I}}}$$

Derivative of determinant

$\psi(\underline{\underline{A}}) = \det(\underline{\underline{A}})$ and if $\underline{\underline{A}}$ is invertible

$$\boxed{D\det(\underline{\underline{A}}) = \det(\underline{\underline{A}})\, \underline{\underline{A}}^{-T}}$$

$\Rightarrow$ for derivation see notes

Time derivative of scaler valued tensor fun.

$\underline{\underline{S}} = \underline{\underline{S}}(t)$ in frame $\{\underline{e}_i\}$

$\underline{\underline{S}}(t) = S_{ij}(t)\, \underline{e}_i \otimes \underline{e}_j$

$$\boxed{\dot{\underline{\underline{S}}} = \frac{d}{dt}\underline{\underline{S}} = \frac{dS_{ij}}{dt}\, \underline{e}_i \otimes \underline{e}_j}$$

How do we take $\frac{d}{dt}\psi(\underline{\underline{S}}(t))$?

By chain rule

$$\frac{d}{dt}\psi(S(t)) = \frac{d}{dt}\psi(S_{11}(t), S_{12}(t), \dots, S_{33}(t))$$

$$= \frac{\partial\psi}{\partial S_{11}}\frac{dS_{11}}{dt} + \dots + \frac{\partial\psi}{\partial S_{33}}\frac{dS_{33}}{dt}$$

$$= \frac{\partial\psi}{\partial S_{ij}}\frac{dS_{ij}}{dt} = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}$$

$\Rightarrow$ chain rule leads to scalar product

$$\boxed{\frac{d}{dt}\psi(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}}$$

Example: $\boxed{\frac{d}{dt}\det(\underline{\underline{S}}(t)) = \det(\underline{\underline{S}})\, \underline{\underline{S}}^{-T} : \dot{\underline{\underline{S}}}}$

Jacobi's formula