Lecture 6: Cauchy Stress Tensor

Logistics: - HW2 graded

→ don't work out identities explicitly

Last time: - Integral theorems

- Derivatives of tensor functions

Today: - Stress

Continuum mass & force concept

continuum is infinitely divisibe

⇒ ignore the discrete nature of atomic structure

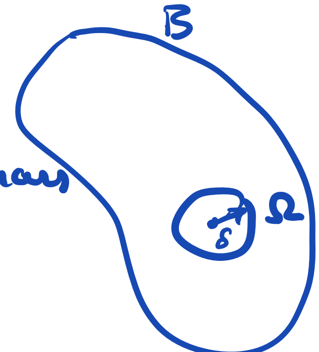
works at length scales bigger than mean atomic spacing.

Mass density

mass is physical property of matter that quantifies its resistance to acceleration

In continuum assumption we assume that mass is continuously distributed

any subset Ω of B with pos. volume has pos. mass



$$V_\Omega = \int_\Omega dV \qquad m_\Omega = \int_\Omega \rho(\underline{x})\, dV$$

where $\rho(\underline{x})$ is mass density field which is defined at any point $\underline{x}$ by

$$\boxed{\rho(\underline{x}) = \lim_{\delta \to 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}}$$

Center of volume: $\underline{x}_V = \frac{1}{V_\Omega} \int_\Omega \underline{x}\, dV$

Center of mass: $\underline{x}_m = \frac{1}{m_\Omega} \int_\Omega \underline{x}\, \rho(\underline{x})\, dV$

Body forces

Forces are the mech. interactions of a body with its environment.

Any force **not** due to physical contact is a body force.

Common body forces: - gravitational field
- electromagnetic field

Inertial/ficticious forces: - centrifugal force
- Coriolis's force

If $\underline{b}$ is a body force field with units

$$\frac{\text{Force}}{\text{Volume}} = \frac{F}{V} = \frac{am}{V} \quad \left[\frac{1}{L^3} \frac{L}{T^2} M = \frac{M}{L^2 T^2} \right]$$

then the resultant force on a body is

$$\boxed{\underline{F}_b = \int_\Omega \underline{b}(\underline{x}) \, dV}$$

and the torque on Ω about point z

$$\boxed{\underline{T}_b = \int_{\underline{\Omega}} (\underline{x} - \underline{z}) \times \underline{b}(x)\, dV}$$

Example: gravitational body force

$$\underline{b}_g = \rho g \quad \left[\frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}\right]$$

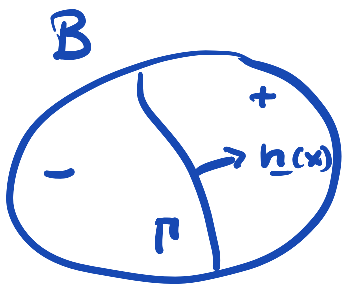
Surface / Contact forces

arise due to contact between bodies

External surface forces act on bounding surface of a body

Internal surface forces act on "imaginary" surfaces within the body

Traction field



arbitrary surface Γ in B with normal $\underline{n}(x)$ that defines pos. & neg. side.

The force per unit area exerted by pos. side onto neg. side is given by traction field $\underline{t}_n$ of Γ

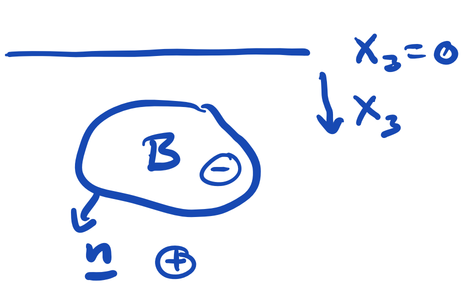
The resultant force due to traction field on Γ

$$\boxed{\underline{F}_s[\Gamma] = \int_\Gamma \underline{t}_n(x)\, dA}$$

The resultant torque about $\underline{z}$ due to traction on Γ

$$\boxed{\underline{T}_s[\Gamma] = \int_\Gamma (\underline{x} - \underline{z}) \times \underline{t}_n(x)\, dA}$$

Example: Archimedes' principle



"Any object, wholly or partially submerged in a fluid is bouyed up by a force equal to the weight of the fluid displaced by the object."

Hydrostatic force field on ∂B

$\underline{t} = -p\,\underline{n}$ where $p = \rho_f g\, x_3$

subst. into def. of resulting force

$$\underline{r}_s[\partial B] = \int_{\partial B} \underline{t}\, dA = \int_{\partial B} -p\,\underline{n}\, dA$$

multiply by a const vect. $\underline{c}$ to apply div theorem

$$\underline{c}\cdot\underline{r}_s[\partial B] = \int_{\partial B} -p\,\underline{c}\cdot\underline{n}\, dA = \int_B -\nabla\cdot(p\,\underline{c})\, dV$$

$$\underline{c}\cdot\underline{r}_s[\partial B] = -\int_B \underline{c}\cdot\nabla p + p\,\cancelto{0}{\nabla\cdot\underline{c}}\,dV$$

$$= \underline{c}\cdot\int_B -\nabla p\,dV$$

which implies

$$\underline{r}_s[\partial B] = \int_B -\nabla p\,dV = -\int \rho g\,\underline{e}_3\,dV$$

$$= -\rho g \int_B dV\,\underline{e}_3$$

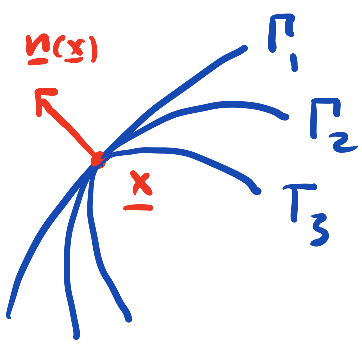
$$= -\rho_f g V_B\,\underline{e}_3 = -M_f g\,\underline{e}_3 = -W_f\,\underline{e}_3$$

can also show that the resulting torque $\underline{\tau}_s[\partial B]$ is zero at $\underline{x}_v$ (cent. of vol.)

$\Rightarrow$ you can think of buoyancy acting at $\underline{x}_v$

Cauchy's postulate

The traction field $\underline{t}_n(\underline{x})$ on surface Γ in B depends only point wise on unit normal field $\underline{n}(x)$. In particular, there is a traction function s.t $\boxed{\underline{t}_n = \underline{t}_n(\underline{n}(\underline{x}), \underline{x})}$



Assumes that $\underline{t}_n$ is independent of $\nabla \underline{n}$ i.e. the curvature of Γ!

$\Rightarrow$ traction $\underline{t}_i$ on all Γ_i that are tangent at $\underline{x}$ is same $\underline{t}_i = \underline{t}_n$

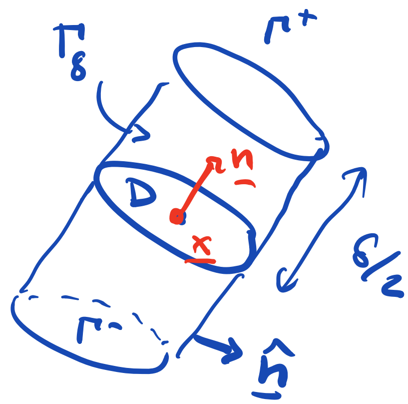
Law of Action and Reaction

If $\underline{t}(\underline{n}, \underline{x})$ is continuous and bounded

$$\boxed{\underline{t}(-\underline{n}, \underline{x}) = -\underline{t}(\underline{n}, \underline{x})}$$

for all $\underline{n}$ and $\underline{x}$.

To show this consider a disk D with arbit. fixed radius around $\underline{x}$.



Let Ω_δ be cylinder with center x, axis $\underline{n}$ and height $\delta > 0$.

End faces Γ^+ & Γ^- mantle Γ_δ

$\hat{\underline{n}}$ is outward normal on Γ_δ

As $\delta \to 0$ then $\Gamma^\pm \to D$ but $\Gamma_\delta \to 0$

$\partial\Omega_\delta = \Gamma^+ \cup \Gamma^- \cup \Gamma_\delta$

$$\lim_{\delta \to 0} \Big[\cancelto{0}{\int_{\Gamma_\delta} \underline{t}(\hat{\underline{n}}(y), y)\, dA} + \int_{\Gamma^+} \underline{t}(\underline{n}, y)\, dA + \int_{\Gamma^-} \underline{t}(-\underline{n}, y)\, dA \Big] = 0$$

first term vanishes because $\underline{t} < \infty$ and

$\Gamma_\delta \to 0$

use $\Gamma^\pm \to D$

$$\int_D \underline{t}((\underline{n}, y) + \underline{t}(-\underline{n}, y)\, dA = 0$$

because D is arbitrary the integrand must be zero.

$$\underline{t}(\underline{n},y) + \underline{t}(-\underline{n},y) = 0$$

$$-\underline{t}(\underline{n},y) = \underline{t}(-\underline{n},y)$$