Lecture 7: Cauchy Stress & Eqbm

Logistics: - HW 3 is due

- no HW this week

Last time: - Mass density $\rho(x) = \lim_{\delta \to 0} \frac{m_{\Omega_\delta}}{V_{\Omega_\delta}}$

- Centers of volume/mass
- Body forces $\underline{b}$ $\frac{\text{Force}}{\text{Vol}}$ ρb $\frac{\text{Force}}{\text{mass}}$
- Surface forces (internal & external)
- resultant force & torque (moment)
- traction field $\underline{t}$
- Cauchy's postulate $\underline{t} = t(n(\underline{x}), x)$
- Action and Reaction $\underline{t}(-\underline{n}, \underline{x}) = -\underline{t}(\underline{n}, \underline{x})$
- Example: Archimedes principle

Today: - Cauchy's theorem → existence of $\underline{\underline{\sigma}}$

- Equilibrium equations
- Stress ellipsoid

Stress tensor

Cauchy's theorem

Let $\underline{t}(\underline{n}, \underline{x})$ be a traction field for body B that satisfies Cauchy's postulate. Then $\underline{t}$ is linear in $\underline{n}$, that is, for any $\underline{x} \in B$ there is a sec. ord tensor field $\underline{\underline{\sigma}}(\underline{x}) \in \mathcal{V}^2$ s.t.

$$\boxed{\underline{t}(\underline{n}, \underline{x}) = \underline{\underline{\sigma}}(\underline{x})\,\underline{n}}$$

where $\underline{\underline{\sigma}}(\underline{x})$ is the Cauchy stress field.

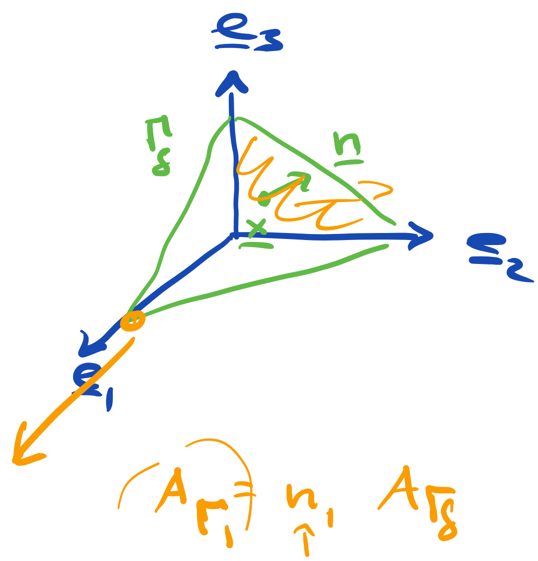
Frame $\{\underline{e}_i\}$

point $\underline{x}$ with normal $\underline{n}$

s.t $\underline{n} \cdot \underline{e}_i \geq 0$

Γ_δ is triangle with center $\underline{x}$ & normal $\underline{n}$

max. edge length is δ



Ω_δ tetrahedron bounded by Γ_δ
and the coordinate planes Γ_i
with outward normal $\underline{n}_j = -\underline{e}_j$

$$\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \int_{\partial\Omega_\delta} \underline{t}(\underline{n}, \underline{x})\, dA = 0$$

$$\lim_{\delta \to 0} \frac{1}{A_{\partial\Omega_\delta}} \left[\int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{y})\, dA + \sum_{j=1}^{3} \int_{\Gamma_j} \underline{t}(-\underline{e}_j, \underline{y})\, dA \right] = 0$$

Each Γ_j can be linearly mapped into Γ_δ
with const Jacobian.

$$n_j = \underline{n} \cdot \underline{e}_j > 0 \Rightarrow \overline{A_{\Gamma_j}} = n_j A_{\Gamma_\delta}$$

$$\Rightarrow dA_{\Gamma_j} = n_j\, dA_{\Gamma_\delta}$$

$$A_{\partial\Omega_\delta} = A_{\Gamma_\delta} + \sum_{j=1}^{3} A_{\Gamma_j} = \lambda A_{\Gamma_\delta} \qquad \lambda = 1 + \sum_{j=1}^{3} n_j$$

substitute

$$\lim_{\delta \to 0} \frac{1}{\lambda A_{\Gamma_\delta}} \int_{\Gamma_\delta} \underline{t}(\underline{n}, \underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j, \underline{x})\, n_j\, dA = 0$$

as $\delta \to 0$ $\quad \Gamma_\delta$ shrinks to $\underline{x}$

by mean value Thm for integrals

the limit is given by integrand

$$\underline{t}(\underline{n},\underline{x}) + \sum_{j=1}^{3} \underline{t}(-\underline{e}_j,\underline{x})\, n_j = 0$$

$$\underline{t}(\underline{n},\underline{x}) = -\sum_{j=1}^{3} \underline{t}(-\underline{e}_j,\underline{x})\, n_j$$

$$\underline{t}(\underline{n},\underline{x}) = \sum_{j=1}^{3} \underline{t}(\underline{e}_j,\underline{x})\, n_j$$

summation convention

$$\underline{t}(\underline{n},\underline{x}) = \underline{t}(\underline{e}_j,\underline{x})\, n_j$$

$n_j = \underline{n} \cdot \underline{e}_j$

$$= (\underline{e}_j \cdot \underline{n})\, \underline{t}(\underline{e}_j,\underline{x})$$

$(\underline{a} \otimes \underline{b})\underline{c} = (\underline{b} \cdot \underline{c})\, \underline{a}$

$$= \big(\underbrace{\underline{t}(\underline{e}_j,\underline{x}) \otimes \underline{e}_j}\big)\, \underline{n}$$

$$= \underline{\underline{\sigma}}(\underline{x})\, n$$

$\Rightarrow \quad \underline{\underline{\sigma}} = \underline{t}(\underline{e}_j,\underline{x}) \otimes \underline{e}_j$

write $\underline{t}(\underline{e}_j,\underline{x}) = t_i(\underline{e}_j,\underline{x})\, \underline{e}_i$

subst

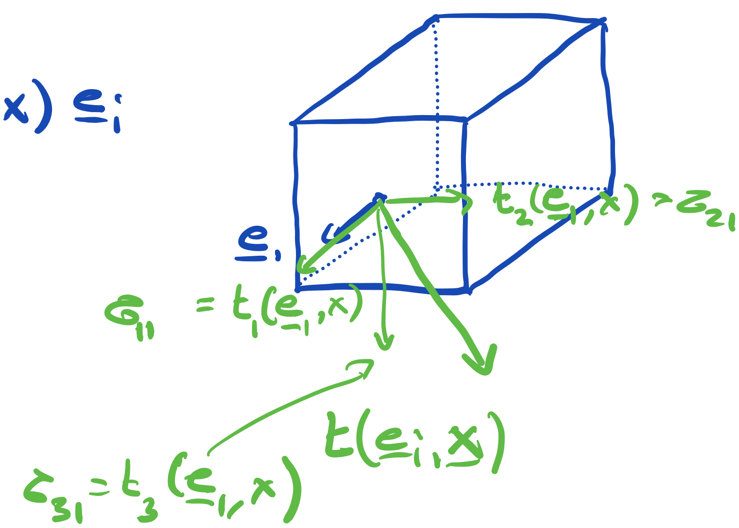
$$\underline{\underline{\sigma}}(x) = \underbrace{t_i(\underline{e}_j, \underline{x})}_{\sigma_{ij}} \, \underline{e}_i \otimes \underline{e}_j$$

definition of Cauchy stress tensor

$$\boxed{\underline{\underline{\sigma}} = \sigma_{ij} \, \underline{e}_i \otimes \underline{e}_j \qquad \sigma_{ij} = t_i(\underline{e}_j, \underline{x})}$$

Hence σ_{ij} is the i-th component of the traction on the j-th coordinate plane.

$$\underline{t}(\underline{e}_1, \underline{x}) = t_i(\underline{e}_1, x)\, \underline{e}_i$$
$$= \sigma_{i1} \underline{e}_i$$



$$\underline{t} = \underline{\underline{\sigma}} \; \underline{n}$$
$$= \sigma_{ij} \, n_j \, \underline{e}_i$$

Mechanical equilibrium

Consider body B under influence of const body force $\rho \underline{b}$ and an external traction $\underline{t}$

Necessary cond. for eqbm
the resultant force and torque vanish for every subset $\Omega \subset B$.

$$\boxed{\begin{aligned} \underline{r}[\Omega] &= \underline{r}_b[\Omega] + \underline{r}_s[\Omega] = \int_\Omega \rho \underline{b}\, dV + \int_{\partial\Omega} \underline{t}\, dA = 0 \\ \underline{\tau}[\Omega] &= \underline{\tau}_b[\Omega] + \underline{\tau}_s[\Omega] = \int_\Omega \underline{x} \times \rho b\, dV + \int_{\partial\Omega} \underline{x} \times \underline{t}\, dA = 0 \end{aligned}}$$

for any Ω

If $\underline{r}[\Omega] = 0$ then $\underline{\tau}[\Omega]$ is independent of $z \Rightarrow (\underline{x} - \underline{z}) \times \underline{t} = \underline{x} \times \underline{t}$

Local mech. equilibrium equations

If $\underline{\underline{\sigma}}(\underline{x})$ is differentiable

$\underline{b}$ and ρ are continuous

the eqbm imply

$$\boxed{\begin{rcases} \nabla\cdot\underline{\underline{\sigma}}(\underline{x}) + \rho(\underline{x})\,\underline{b}(x) = 0 \\ \underline{\underline{\sigma}}^T(\underline{x}) = \underline{\underline{\sigma}}(x) \end{rcases} \text{ for all } \underline{x}\in B}$$

in components

$$\sigma_{ij,j} + \rho b_i = 0$$

$$\sigma_{ij} = \sigma_{ji}$$

Substitute, $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$ into $\underline{r}[\Omega]$

$$\underline{r}[\Omega] = \int_{\partial\Omega} \underline{\underline{\sigma}}\,\underline{n}\, dA + \int_{\Omega} \rho\underline{b}\, dV = 0$$

use Tensor div. thm

$$\int_{\Omega} \nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b}\; dV = 0$$

by the arbitrariness of $\Omega \Rightarrow$

$$\Rightarrow \quad \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0$$

To establish symmetry of σ subst. into $\underline{\tau}[\Omega]$

$$\underline{\tau}[\Omega] = \int_{\partial\Omega} \underline{x} \times \underline{t} \quad dA + \int_{\Omega} \underline{x} \times \rho \underline{b} \, dV$$

$$\int_{\partial\Omega} \underline{x} \times (\underline{\underline{\sigma}} \, \underline{n}) \, dA - \int \underline{x} \times \nabla \cdot \underline{\underline{\sigma}} \, dV = 0$$

need to turn l.h.s into $\int_{\Omega}$

rewrite $\quad \underline{x} \times (\underline{\underline{\sigma}} \, \underline{n}) = \underline{\underline{R}} \, \underline{n}$

where $\quad R_{il} = \epsilon_{ijk} x_j \sigma_{kl}$

$$\int_{\partial\Omega} \underline{\underline{R}} n \, dA - \int_{\Omega} \underline{x} \times \nabla \cdot \underline{\underline{\sigma}} \, dV = 0$$

using tensor div. thm $\quad \int_{\partial\Omega} \underline{\underline{R}} n \, dA = \int_{\Omega} \nabla \cdot \underline{\underline{R}} \, dV$

$$\longrightarrow \int_{\Omega} \nabla \cdot \underline{\underline{R}} - \underline{x} \times \nabla \cdot \underline{\underline{\sigma}} \, dV = 0$$

in components

$$\underbrace{(\epsilon_{ijk}\, x_j\, \sigma_{kl})}_{R_{il,l}}{}_{,l} - \epsilon_{ijk}\, x_j\, \sigma_{kl,l}$$

chain rule

$$\epsilon_{ijk}\, x_{j,l}\, \sigma_{kl} + \cancel{\epsilon_{ijk}\, x_j\, \sigma_{kl,l}} - \cancel{\epsilon_{ijk}\, x_j\, \sigma_{kl,l}} = 0$$

$$\epsilon_{ijk}\, x_{j,l}\, \sigma_{kl} = \epsilon_{ijk}\, \delta_{jl}\, \sigma_{kl} = \epsilon_{ijk}\, \sigma_{kj} = 0$$

If $\epsilon_{ijk}\sigma_{kj} = 0$ then $\epsilon_{ikj}\sigma_{jk} = 0$
because j & k are dummy's.

$$\epsilon_{ijk}\sigma_{kj} + \epsilon_{ikj}\sigma_{jk} = 0$$

$$\epsilon_{ijk}(\sigma_{kj} - \sigma_{jk}) = 0$$

we can i separate from j & k

$$\sigma_{kj} - \sigma_{jk} = 0$$

$$\Rightarrow \sigma_{kj} = \sigma_{jk} \qquad \underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$$

local Eqbm equs

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0 \qquad 3 \text{ eqn}$$

$$\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \qquad 3 \text{ eqn}$$

$\left.\right\}$ 6 eqn

$\sigma_{12} = \sigma_{21} \quad \sigma_{13} = \sigma_{31} \quad \sigma_{23} = \sigma_{32}$

but $\underline{\underline{\sigma}}$ has 9 unknown components

$\Rightarrow$ need a constitutive law

$\underline{\underline{\sigma}} \rightarrow \underline{u}$ displacement

$\underline{v}$ velocity