Lecture 8: Normal and shear stress

Logistics: - no hw this week

- no office hrs unless requested

Last time: • Cauchy stress tensor $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$

$$\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \otimes \underline{e}_j \qquad \sigma_{ij} = t_i(\underline{e}_j, \underline{x})$$

• Equilibrium Equations

$$\underline{r} = 0 \rightarrow \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0$$

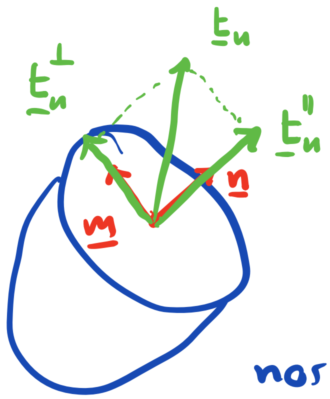
$$\underline{\tau} = 0 \rightarrow \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$$

6 eqns but 9 unknowns!

Today: - Normal & shear stress

- principal stresses
- Stress ellipsoid
- Simple states of stress

Normal & Shear stresses



Projection matrices:

$$\underline{\underline{P}}^{\parallel} = \underline{n} \otimes \underline{n}$$

$$\underline{\underline{P}}^{\perp} = \underline{\underline{I}} - \underline{n} \otimes \underline{n} = \underline{m} \otimes \underline{m}$$

normal stress: $\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel}\, \underline{t}_n = (\underline{n} \cdot \underline{t}_n)\, \underline{n} = \sigma_n\, \underline{n}$

shear stress: $\underline{t}_n^{\perp} = \underline{\underline{P}}^{\perp}\, \underline{t}_n = (\underline{m} \cdot \underline{t}_n)\, \underline{m} = \tau\, \underline{m}$

Magnitudes

$\sigma_n = \underline{n} \cdot \underline{t}_n = \underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n}$ or $\sigma_n = n_i\, \sigma_{ij}\, n_j$

$\tau = \underline{m} \cdot t_n = \underline{m} \cdot \underline{\underline{\sigma}}\, \underline{n}$ or $\tau = m_i\, \sigma_{ij}\, n_j$

$\sigma_n > 0 \Rightarrow$ stress is tensile

$\sigma_n < 0 \Rightarrow$ stress is compressive

From geometry: $\underline{t}_n = \underline{t}_n^{\parallel} + \underline{t}_n^{\perp}$

$$\boxed{|\underline{t}_n|^2 = \sigma_n^2 + \tau^2}$$

Extremal stress values

Given $\underline{\underline{\sigma}}$ at $\underline{x}$ what unit normals $\underline{n}$ corresponding to min & max normal stress σ_n

$\Rightarrow$ constraind optimization problem

need to find max and min $\sigma_n = \sigma_n(\underline{n})$

but with constraint $|\underline{n}| = 1$

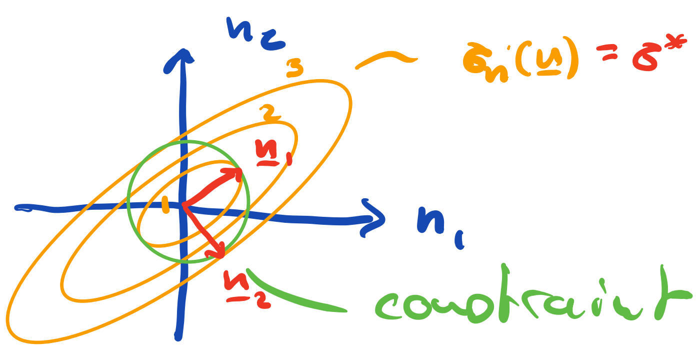
Lagrange multiplier method

$$\mathcal{L}(\underline{n},\lambda) = \underbrace{\underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n}}_{\substack{\sigma_n(\underline{n}) \\ \text{function} \\ \text{to optimize}}} - \underbrace{\lambda}_{\text{Lag. multiplier}}\,(\underbrace{\underline{n}\cdot\underline{n} - 1}_{\text{constraint}})$$

$$= n_i\,\sigma_{ij}\,n_j - \lambda\,(n_i n_i - 1)$$

For visualization assume $\underline{\underline{\sigma}}$ is spd (compression)

& $\sigma_n = \underline{n}\cdot\underline{\underline{\sigma}}\,\underline{n} \rightarrow$ is quadratic



To find extremal values we need to find stationary points of $\mathcal{L}(\underline{n},\lambda)$

$$\mathcal{L} = n_i \sigma_{ij} n_j - \lambda (n_i n_i - 1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_i n_i - 1 = 0 \quad \rightarrow \quad |\underline{n}| = 1$$

$$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij} (n_{i,k} n_j + n_i n_{j,k}) - \lambda (n_{i,k} n_i + n_i n_{i,k})$$

$$= \sigma_{ij} (\delta_{ik} n_j + \delta_{jk} n_i) - 2\lambda \delta_{ik} n_i$$

$$= \sigma_{kj} n_j + \overset{= \sigma_{ki}}{\sigma_{ik}} n_i - 2\lambda n_k$$

$$= 2(\sigma_{ik} n_k - \lambda n_k) = 0$$

In symbolic notation

$$\underline{\underline{\sigma}}\, \underline{n} - \lambda \underline{n} = 0 \quad \& \quad |\underline{n}| = 1$$

$$\Rightarrow \boxed{(\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\, \underline{n} = 0} \quad \text{eigenvalue problem}$$

Clearly $\underline{n}_i$ directions to max, min normal stress

What is λ?

$$\underline{n} \cdot (\underline{\underline{\sigma}} - \lambda \underline{\underline{I}})\, \underline{n} = 0$$

$$\underbrace{\underline{n} \cdot \underline{\underline{\sigma}}\, \underline{n}}_{\sigma_n = \lambda} - \lambda\, \underline{n} \cdot \underline{n} = 0$$

λ_i's are the principal normal stresses $\lambda_i = \sigma_i$

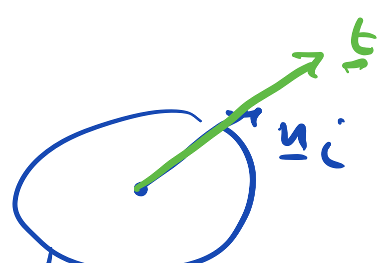
n_i's are the principal directions of $\underline{\underline{\sigma}}$

Since $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ all σ_i's are real and the $\{\underline{n}_i\}$ forms an orthonormal basis

$$\Rightarrow \quad \underline{\underline{\sigma}} = \sum_{i=1}^{3} \sigma_i \, \underline{n}_i \otimes \underline{n}_i$$

Tractions in principal dir.

$$\underline{t}_{\underline{n}_i} = \underline{\underline{\sigma}}\, \underline{n}_i = \sigma_i \, \underline{n}_i$$



$\underline{t}_{\underline{n}_i} \parallel \underline{n}_i \Rightarrow \underline{t}^{\perp}_{\underline{n}_i} = 0$ no shear stress

If σ_i's are distinct $\sigma_1 > \sigma_2 > \sigma_3$ then σ_1 and σ_3 are max & min normal stresses.

Max & min shear stresses

Given principal dir $\{\underline{n}_i\}$ at $\underline{x}$ what is unit vector $\underline{s} = [s_1, s_2, s_3]$ that gives max & min values of shear stress?

$$s_i = \underline{s} \cdot \underline{n}_i$$

In frame $\{n_i\}$ $\underline{t}_s$ is

$$\underline{t}_s = \underline{\underline{\sigma}}\, \underline{s} = \left(\sum_{i=1}^{3} \sigma_i\, n_i \otimes \underline{n}_i\right) \underline{s}$$

$$= \sigma_1 s_1 \underline{n}_1 + \sigma_2 s_2 \underline{n}_2 + \sigma_3 s_3 \underline{n}_3$$

magnitudes of stress on $\underline{s}$

$$t_s = \begin{pmatrix} \sigma_1 s_1 \\ \sigma_2 s_2 \\ \sigma_3 s_3 \end{pmatrix} \quad |t_s|^2 = (\sigma_1 s_1)^2 + (\sigma_2 s_2)^2 + (\sigma_3 s_3)^2$$

$$\sigma_n = \underline{s} \cdot \underline{t}_s = \sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2 = \sigma_i s_i^2$$

$$\tau^2 = |\underline{t}_s|^2 - \sigma_n^2 = (\sigma_1 s_1)^2 + (\sigma_2 s_2)^2 + (\sigma_3 s_3)^2 - (\sigma_1 s_1^2 + \sigma_2 s_2^2 + \sigma_3 s_3^2)^2$$

index notation

$$\boxed{\tau^2 = \sum_{i=1}^{3} \sigma_i^2 s_i^2 - \left(\sum_{i=1}^{3} \sigma_i s_i^2\right)^2}$$

shear stress on some plane with normal $\underline{s}$ in $\{\underline{n}_i\}$

looking for extrema τ^2 under constraint

$$|\underline{s}|^2 - 1 = 0$$

$\Rightarrow$ use Lagrange multipliers

gets ~~wors~~. messy

Here we choose direct elimination.

Eliminate $\underline{s}^2 = 1 \quad s_1^2 + s_2^2 + s_3^2 = 1$

$$s_3^2 = 1 - s_2^2 - s_1^2$$

$$\Rightarrow \tau^2 = \tau^2(s_1, s_2)$$

Just need to find $\frac{\partial \tau^2}{\partial s_1} = \frac{\partial \tau^2}{\partial s_2} = 0$

$$\frac{\partial \tau^2}{\partial s_1} = 2 s_1 (\sigma_1 - \sigma_3)\left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3) s_1^2 + (\sigma_2 - \sigma_3) s_2^2\right]\right\} = 0$$

$$\frac{\partial \tau^2}{\partial s_2} = 2 s_2 (\sigma_2 - \sigma_3)\left\{\boxed{\sigma_2 - \sigma_3 - 2\left[\cancel{(\sigma_1 - \sigma_3) s_1^2} + (\sigma_2 - \sigma_3) s_2^2\right]}\right\} = 0$$

First solution: $s_1 = s_2 = 0 \Rightarrow s_3 = \pm 1 \Rightarrow \underline{s} = \pm \underline{n}_3$

$$\tau^2 = \sigma_3^2 \cdot 1 - (\sigma_3 \cdot 1)^2 = 0$$

$\Rightarrow$ minimum in shear stress

shear stress vanishes on principal planes

analogously this can be shown for $\underline{n}_1, \underline{n}_2$

Second solution: $s_1 = 0$

$$\frac{\partial \tau^2}{\partial s_2} = (\sigma_2 - \sigma_3) - 2\left[(\sigma_2 - \sigma_3) s_2^2\right] = 0$$

$$(\sigma_2 - \sigma_3)\left(\underset{0}{1} - 2 s_2^2\right) = 0 \Rightarrow s_2 = \pm \frac{1}{\sqrt{2}}$$

from $\cancel{s_1^2} + s_2^2 + s_3^2 = 1 \Rightarrow s_3 = \pm \frac{1}{\sqrt{2}}$

$$\underline{s} = \pm \frac{1}{\sqrt{2}} \underline{n}_2 \pm \frac{1}{\sqrt{2}} \underline{n}_3 = \pm \frac{1}{\sqrt{2}} (\underline{n}_2 + \underline{n}_3)$$

$$\tau^2 = \sum \sigma_i^2 s_i^2 - \left(\sum \sigma_i s_i^2\right)^2$$

$$s_1 = 0 \qquad s_2 = s_3 = \pm\frac{1}{\sqrt{2}}$$

$$\tau^2 = \sigma_2^2 \frac{1}{2} + \frac{\sigma_3^2}{2} - \left(\frac{\sigma_2}{2} + \frac{\sigma_3}{2}\right)^2$$

$$= \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} - \frac{\sigma_2^2}{4} - 2\frac{\sigma_2}{2}\frac{\sigma_3}{2} - \frac{\sigma_3^2}{4}$$

$$= \frac{\sigma_2^2}{4} - 2\frac{\sigma_2 \, \sigma_3}{2 \;\; 2} + \frac{\sigma_3^2}{4}$$

$$= \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

We have following two solutions

min.: $\tau = 0$ for $\underline{s} = \pm \underline{n}_3$

max.: $\tau = \frac{1}{2}(\sigma_2 - \sigma_3)$ for $\underline{s} = \pm\frac{\underline{n}_2}{\sqrt{2}} \pm \frac{\underline{n}_3}{\sqrt{2}}$

$\Rightarrow$ similarly we can find two additional pairs of solns by eliminating s_2 and s_1, and following similar steps

Minimum shear stresses

$$\boxed{\tau = 0 \quad \text{on} \quad \underline{s} = \pm \underline{n}_1 \quad \underline{s} = \pm \underline{n}_2 \quad \underline{s} = \pm \underline{n}_3}$$

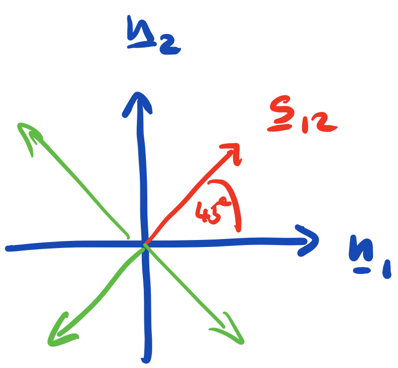
Maximum shear stress

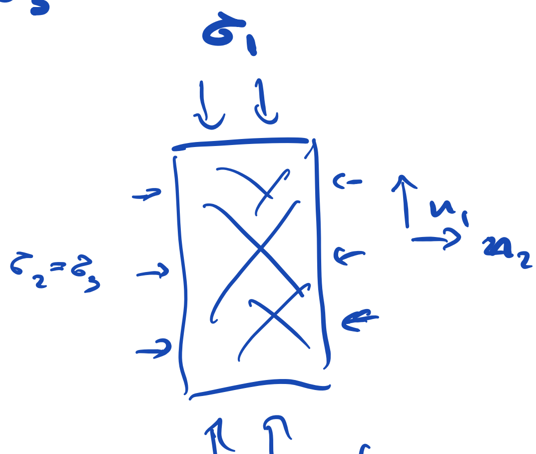
$$\tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3) \quad \text{on} \quad \underline{s}_{23} = \frac{1}{\sqrt{2}}(\pm \underline{n}_2 \pm \underline{n}_3)$$

$$\tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3) \quad \text{on} \quad \underline{s}_{13} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_3)$$

$$\tau_{12} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad \text{on} \quad \underline{s}_{12} = \frac{1}{\sqrt{2}}(\pm \underline{n}_1 \pm \underline{n}_2)$$

assume $\sigma_1 \geq \sigma_2 \geq \sigma_3$





$\Rightarrow$ sets of conjugate shear fractures at $45°$ to σ_1