Lecture 9: Mohr circle & failure

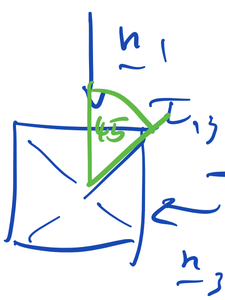
Logistics: - HW 4 posted today

Last time: - Normal & shear stress $\quad t_n = \underline{\sigma}\, \underline{n}$

$$\underline{t}_n^{\parallel} = \underline{\underline{P}}^{\parallel} \underline{n} \qquad \underline{t}^{\perp} = \underline{\underline{P}}^{\perp} \underline{n}$$

$$\sigma_n = |\underline{t}_n| \qquad \tau = |\underline{t}^{\perp}|$$

- Extremal values of normal stress



$$\Rightarrow (\underline{\underline{\sigma}} - \sigma_i \underline{\underline{I}})\, \underline{n}_i = 0 \qquad \sigma_1 > \sigma_2 > \sigma_3$$

- Extremal values of shear stress

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} (\pm \underline{n}_1 \pm \underline{n}_2)$$

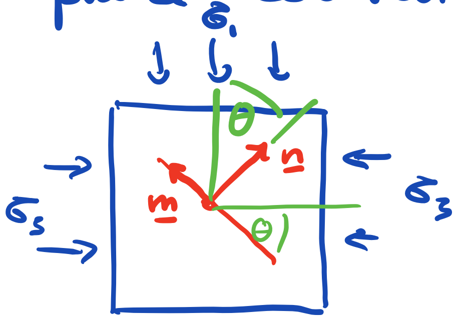
Today:
- Mohr circle
- Failure criteria
- simple states of stress
- stress ellipsoid

Mohr circle

graprcal way of displaying the normal and shear stresse on all planes through a point.

For simplicity → 2D case

plane containin $\underline{n}_1$ & $\underline{n}_3$



$$\underline{n} \cdot \underline{n}_1 = \cos\theta$$

$$\underline{n} \cdot \underline{n}_3 = \sin\theta$$

$$\underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \underline{m} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Stress in principal frame:

$$\underline{\underline{\sigma}} = \sigma_1 \, \underline{n}_1 \otimes \underline{n}_1 + \cancel{\sigma_2 \, \underline{n}_2 \otimes \underline{n}_2} + \sigma_3 \, \underline{n}_3 \otimes \underline{n}_3$$

traction: $\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \sigma_1 \cos\theta \, \underline{n}_1 + \sigma_3 \sin\theta \, \underline{n}_3$

normal stress: $\sigma_n = \underline{n} \cdot \underline{t}_n = \sigma_1 \cos^2\theta + \sigma_3 \sin^2\theta$

use $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\boxed{\sigma = \underbrace{\frac{\sigma_1+\sigma_3}{2}}_{\text{mean stress}} + \underbrace{\frac{\sigma_1-\sigma_3}{2}}_{\text{max shear stress } \tau_{13}} \cos 2\theta}$$

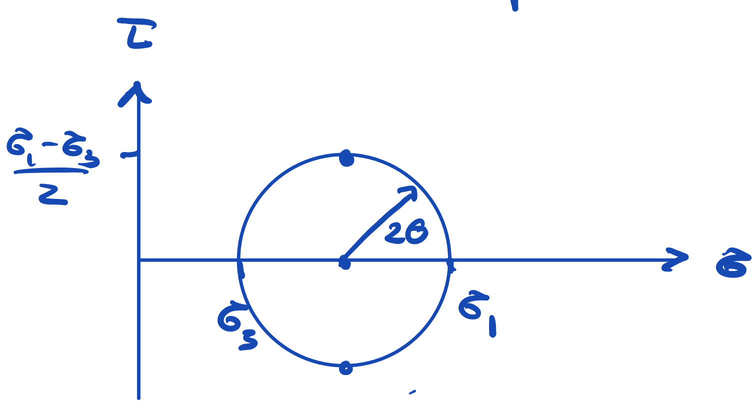
shear stress: $\tau = \underline{m} \cdot \underline{t}_n = (\sigma_1 - \sigma_3)\sin\theta\cos\theta$

use $2\sin\theta\cos\theta = \sin 2\theta$

$$\boxed{\tau = \frac{\sigma_1 - \sigma_3}{2}\sin 2\theta}$$

Equ for circle in $\tau\sigma$ space with radius $R = \frac{\sigma_1 - \sigma_3}{2}$ with origin $\left(\frac{\sigma_1 + \sigma_3}{2}, 0\right)$

For Mohr circle compressive stresses are assumed positive.

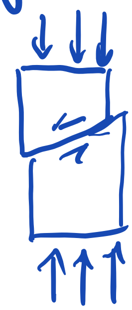


Mohr circle

$\Rightarrow$ Experiments show failure does not occur on surfaces corresponding to max shear stress

Failure criteria for shear fracture

shear fracture is the most common type of brittle failure.

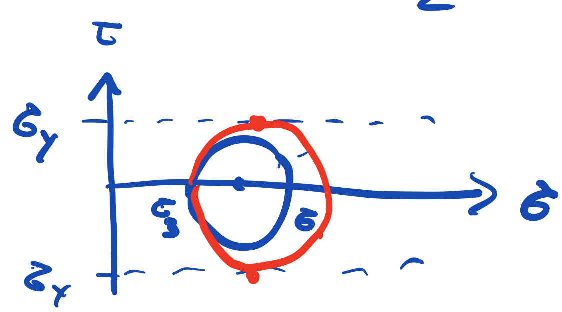


⇒ Empirical criterion for prediction of shear failure.

I) Tresca criterion

Fracture occurs when max shear stress reaches the shear strength σ_Y

$$|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2} = \sigma_Y$$



this predicts failure along 45° max shear planes.

II Coulomb failure criterion

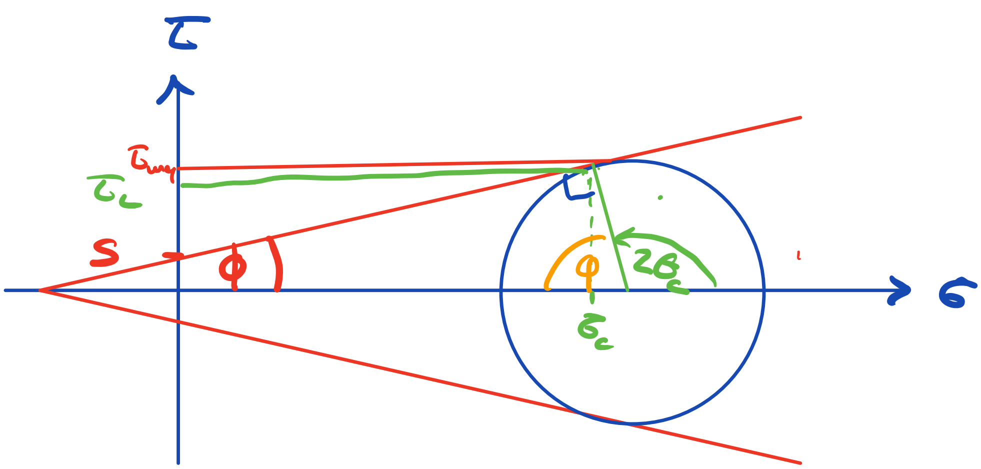
Shear fracture also depends on normal stress

$$\boxed{|\tau| = S + \mu' \sigma}$$

S = cohesive strength $\sim 10-100$ MPa

$\mu' = \tan\phi$ internal friction ~ 0.6

ϕ = angle of internal friction $\sim 30°$

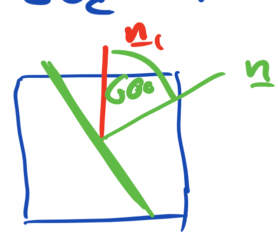


angle of failure

$$\phi + \frac{\pi}{2} + \varphi = \pi \qquad \varphi + 2\theta_c = \pi$$

$$\phi + \frac{\pi}{2} + (\pi - 2\theta_c) = \pi$$

$$\theta_c = \underset{90° +}{\frac{\pi}{4}} + \underset{15°}{\frac{\phi}{2}} \Rightarrow 60°$$



Byerlee's law (Amonton's law)

All brittle rocks already contain preexisting fractures and fail by reactivating them

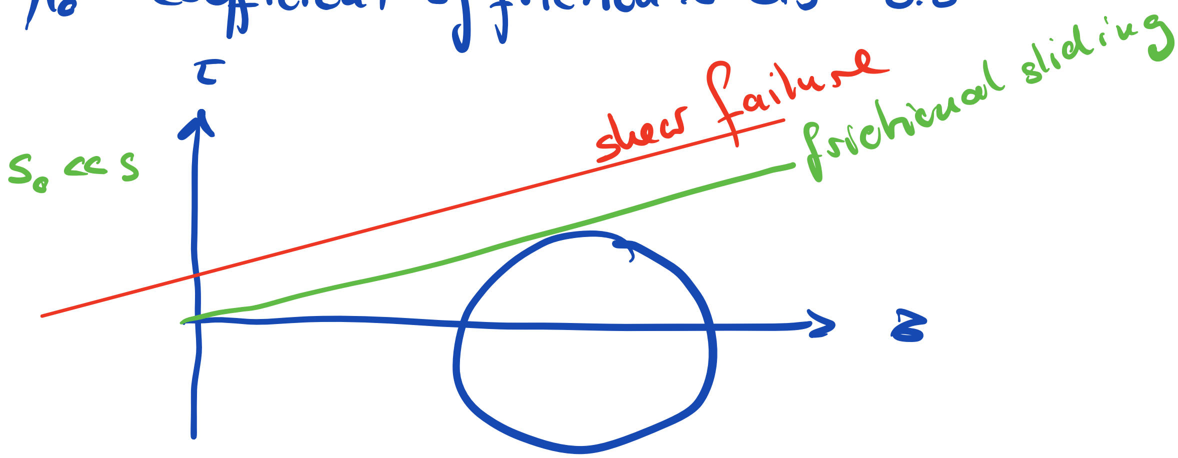
⇒ fail by friction.

Criterion for frictional sliding

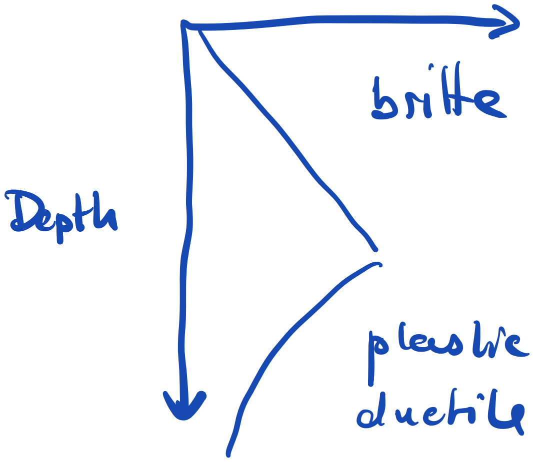
$$|\tau| = S_0 + \mu_0 \sigma$$

S_0 = cohesion of fault ~ 1-10 MPa

μ_0 = coefficient of friction ~ 0.5 - 0.8



⇒ strength of brittle rocks is determined by frictional sliding

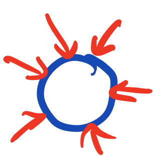


rocks get stronger with depth due to increase in mean ~~&~~ normal stress.

Simple states of stress

I) Hydrostatic stress

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} \qquad \underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = -p\,\underline{n}$$

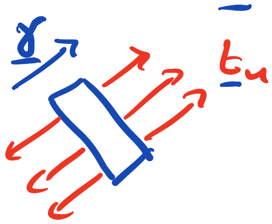


$$\sigma = |\underline{t}_n^{\parallel}| = \underline{t}_n \cdot \underline{n} = -p = \sigma_1 = \sigma_2 = \sigma_3$$

$$\tau = |\underline{t}_n^{\perp}| = 0$$ no shear stresses

II Uniaxial stress

$$\underline{\underline{\sigma}} = \sigma\, \underline{\gamma} \otimes \underline{\gamma} \qquad |\underline{\gamma}| = 1$$



$$\underline{t}_n = \underline{\underline{\sigma}}\, n = \sigma(\underline{\gamma} \otimes \underline{\gamma})\,\underline{n} = \sigma(\underline{n}\cdot\underline{\gamma})\,\underline{\gamma}$$

traction always parallel $\underline{\gamma}$

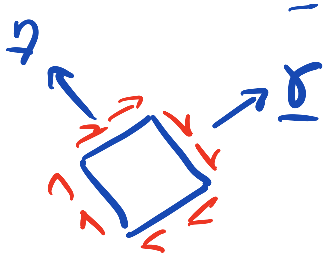
$\sigma > 0$ pure tension

$\sigma < 0$ pure compression

III Pure shear stress

$\underline{\gamma} \cdot \underline{\eta} = 0 \qquad |\underline{\gamma}| = |\underline{\eta}| = 1$

$$\underline{\underline{\sigma}} = \tau\,(\underline{\gamma} \otimes \underline{\eta} + \underline{\eta} \otimes \underline{\gamma})$$



$$\underline{t}_n = \underline{\underline{\sigma}}\,\underline{n} = \tau\,(\underline{\eta} \cdot \underline{n})\,\underline{\gamma} + \tau\,(\underline{\gamma} \cdot \underline{n})\,\underline{\eta}$$

$\underline{n} = \underline{\gamma}$: $\quad \underline{t}_n = \tau\,\underline{\eta}$

$\underline{n} = \underline{\eta}$ $\quad \underline{t}_n = \tau\,\underline{\gamma}$

IV Uniform/simple shear

$$\sigma_{12} = \sigma_{21} = \tau$$

$$\underline{\underline{\sigma}} = \tau\,(\underline{n}_1 \otimes \underline{n}_2 + \underline{n}_2 \otimes \underline{n}_1) = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

V Plane stress

If there exists a $\underline{\gamma}$ and $\underline{\eta}$ $(\underline{\gamma} \cdot \underline{\eta} = 0)$ such that the matrix respresentation of $\underline{\underline{\sigma}}$ in frame $\{\underline{\gamma}, \underline{\eta}, \underline{\gamma} \times \underline{\eta}\}$ is of the form

$$[\underline{\underline{\sigma}}] = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then a state of plane stress exists.

Spherical & deviatoric stress tensors

We can write Cauchy stress as

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_s + \underline{\underline{\sigma}}_D$$

spherical stress tensor: $\underline{\underline{\sigma}}_s = -p\underline{\underline{I}}$ $\quad p = -\frac{1}{3}tr(\underline{\underline{\sigma}})$

$= tr(\underline{\underline{\sigma}})$

deviatoric stress tensor: $\underline{\underline{\sigma}}_D = \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_s = \underline{\underline{\sigma}} + p\underline{\underline{I}}$

Pressure is the mean normal stress

$$p = -\frac{1}{3}tr(\sigma) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Marc will check $\frac{1}{3}$

Spherical part of $\underline{\underline{\sigma}}$ is part that changes volume of body

deviatoric part changes shape of body

Many constitutive laws are based on invariants of deviatoric stress.

$I_1(\underline{\underline{\sigma}}_D) = tr(\underline{\underline{\sigma}}_D) = 0$

$J_2(\underline{\underline{\sigma}}_D) = -I_2(\underline{\underline{\sigma}}_D) = \frac{1}{2}\, \underline{\underline{\sigma}}_D : \underline{\underline{\sigma}}_D$

$J_3(\underline{\underline{\sigma}}_D) = I_3(\underline{\underline{\sigma}}_D) = det(\underline{\underline{\sigma}}_D)$