Isothermal Fluid Mechanics

→ application of Eulerian balance laws

→ neglect thermal effects.

10 equations:

$\underline{v}_m = \dot{\underline{\varphi}}$ 3 kinematic

$\frac{\partial \rho}{\partial t} + \nabla_{\underline{x}} \cdot (\rho \underline{v}) = 0$ 1 mass balance

$\rho \dot{\underline{v}} - \nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} = \rho b$ 3 linear mom.

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ 3 angular mom.

16 unknown quantities

$\underline{\varphi} \quad \underline{v} \quad \rho \quad \underline{\underline{\sigma}}$ $3 + 3 + 1 + 9 = 16$

Constitutive equ that relates 6 independent components of $\underline{\underline{\sigma}}$ to $\underline{\varphi}, \underline{v}, \rho$.

If there is a material constraint this adds both an equation $\gamma = 0$ and an unknown q.

Ideal Fluids

A fluid is ideal if

1) Uniform reference mass density: $\rho_0(\underline{X}) = \rho_0 > 0$

2) Fluid is incompressible: $\nabla_x \cdot \underline{v} = 0$

3) Cauchy stress is spherical: $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}$

$\Rightarrow$ no shear stresses $\quad \underline{t} = \underline{\underline{\sigma}}\, n = -p\,\underline{n}$

$1 + 2 \Rightarrow \quad \rho(x,t) = \rho_0$

substituting into mass balance

$$\cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \nabla_x \cdot (\rho_0 \underline{v}) = 0 \quad \Rightarrow \quad \nabla_x \cdot \underline{v} = 0$$

substituting into mom. balance

$$\rho_0 \dot{\underline{v}} = \nabla_x \cdot (-p\,\underline{\underline{I}}) + \rho_0 \underline{b}$$

where $\dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\,\underline{v}$ and $\nabla_x \cdot (-p\,\underline{\underline{I}}) = -\nabla_x p$

we obtain closed system for $\underline{v}$ and p

$$\boxed{\begin{aligned} \rho_0\left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\,\underline{v}\right) &= -\nabla_x p + \rho_0 \underline{b} \\ \nabla_x \cdot \underline{v} &= 0 \end{aligned}}$$

Euler Equations

Note: p has undetermined constant.

Frame Indifference of Euler Equations

The stress field of an ideal fluid is entirely reactive

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^r + \underline{\underline{\sigma}}^a = -p\underline{\underline{I}}$$

$$\Rightarrow \quad \underline{\underline{\sigma}}^r = -p\,\underline{\underline{I}} \qquad \underline{\underline{\sigma}}^a = \underline{\underline{0}}$$

where p is the multiplier associated with the incompressibility constraint. $\nabla_x \cdot \underline{v} = 0$

For a constrained model to be frame indifferent both the constraint $\gamma(\underline{\underline{F}}(\underline{X},t)) = 0$ and the active stress $\underline{\underline{\sigma}}^a$ must be frame indifferent.

Assuming a superposed rigid motion $\underline{x}^* = \underline{\underline{Q}}\,\underline{x} + \underline{c}$

- $\gamma(\underline{\underline{F}}^*) = \det(\underline{\underline{F}}^*) - 1 = \cancel{\det(\underline{\underline{Q}})}^{1}\det(\underline{\underline{F}}) - 1 = \gamma(\underline{\underline{F}})$

  $\Rightarrow$ constraint field is indifferent

- $\underline{\underline{\sigma}}^a = \underline{\underline{0}}$ trivially indifferent

$\Rightarrow$ material model for an ideal fluid is frame-indif.

Mechanical energy considerations

Even in an isothermal model the entropy inequality provides a constraint on material model

$$\boxed{\rho \dot{\psi} \leq \underline{\underline{\sigma}} : \underline{\underline{d}}} \qquad \text{from Lecture 18}$$

also called Mechanical Energy Inequality (MEI)

where $\underline{\underline{\sigma}} : \underline{\underline{d}} = -p\underline{\underline{I}} : \text{sym}(\nabla_x \underline{v})$

using $\underline{\underline{I}} : \underline{\underline{A}} = \text{tr}(\underline{\underline{A}})$ and $\text{tr}(\text{sym}(\underline{\underline{A}})) = \text{tr}(\underline{\underline{A}})$

we have $\underline{\underline{\sigma}} : \underline{\underline{d}} = -p\,\text{tr}(\nabla_x \underline{v}) = -p\,\nabla_x \cdot \underline{v} = 0$

$\Rightarrow$ stress power is zero

$\Rightarrow$ consistent with MEI

Bernoulli streamline theorem (steady)

From PS 4 : $(\nabla_x \underline{v})\underline{v} = (\nabla \times \underline{v}) \times \underline{v} + \frac{1}{2}\nabla |\underline{v}|^2$

subst. into Euler equation

$$\frac{\partial v}{\partial t} + (\nabla_x \times \underline{v}) \times \underline{v} = -\frac{1}{2}\nabla_x |\underline{v}|^2 - \frac{1}{\rho_0}\nabla_x p + \underline{b}$$

for conservative body force $\underline{b} = -\nabla \psi$

$$\frac{\partial \underline{v}}{\partial t} + (\nabla_x \times \underline{v}) \times \underline{v} = -\nabla_x\left(\frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + \psi\right) = -\nabla_x H$$

$H = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + \psi$ where $\psi = gz$ for gravity

H has units of energy/mass

$E_k = \frac{1}{2}m|\underline{v}|^2$ $\quad E_G = mgz$ $\quad E_E = m\int_{p_0}^{p}\frac{dp}{\rho} = m\frac{p - \cancelto{0}{p_0}}{\rho_0}$

$$\boxed{H = \frac{E}{m} = \frac{E_k}{m} + \frac{E_E}{m} + \frac{E_G}{m} = \frac{1}{2}|\underline{v}|^2 + \frac{p}{\rho_0} + gz}$$

Steady flow

$$(\nabla_x \times \underline{v}) \times \underline{v} = -\nabla_x H$$

take dot product from left

$$\cancelto{0}{\underline{v}\cdot(\underset{\substack{\uparrow \\ \text{perp. to } \underline{v}}}{\nabla_x \times \underline{v}}) \times \underline{v}} = -\underline{v}\cdot\nabla_x H$$

$\Rightarrow$ $\boxed{\underline{v}\cdot\nabla_x H = 0}$ Bernoulli's Thm for steady flow

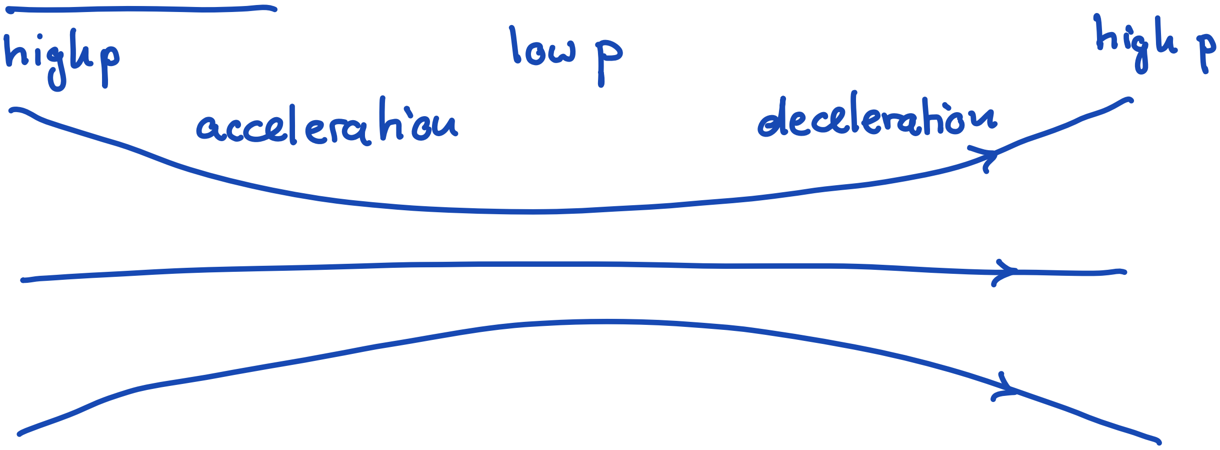
implies that H is <u>constant</u> along a streamline

Stream line is integral curve of $\underline{v}$.

$\Rightarrow$ Energy is conserved in ideal flow

because $\rho\dot{\psi} = \underline{\underline{\sigma}} : \underline{\underline{d}} = 0$.

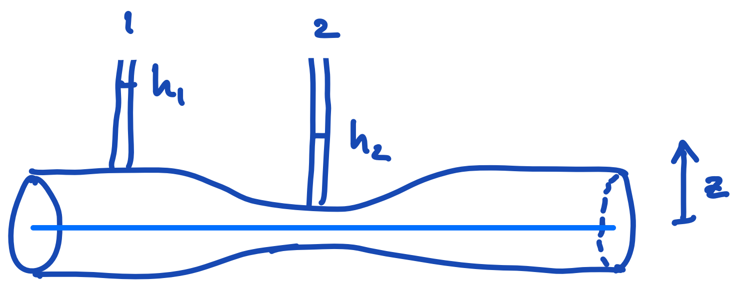
Intuition



Examples:

1) Venturi meter

H is const along



$$H = \frac{p_1}{\rho} + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + \frac{1}{2} v_2^2 \qquad (z=0)$$

$$A_1 v_1 = A_2 v_2 \qquad \text{(mass cons.)} \qquad v_2 = \frac{A_1}{A_2} v_1$$

$$p_1 - p_2 = \frac{\rho}{2}\left(v_2^2 - v_1^2\right) = \frac{\rho}{2}\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right) v_1^2$$

hydrostatics: $p_1 - p_0 = \rho g h_1 \qquad p_2 - p_0 = \rho g h_2$

$$\rho g (h_1 - h_2) = \frac{\rho}{2}\left(\frac{A_1^2}{A_2^2} - 1\right) v_1^2$$

Measure v_1 as:

$$\boxed{v_1^2 = \frac{2g(h_1 - h_2)}{(A_1^2/A_2^2 - 1)}}$$

Irrotational Motion

A velocity field $\underline{v}$ with spin field $\underline{\underline{W}} = \text{skew}(\nabla_x \underline{v})$ is irrotational if

$$\underline{\underline{W}}(\underline{x},t) = \underline{\underline{0}} \quad \text{or} \quad \nabla_x \times \underline{v} = \underline{\omega} = 0$$

During an irrotational motion material particles experience no net rotation.

Velocity potential

Helmholtz decomposition of velocity

$$\underline{v} = -\nabla_x \phi + \nabla_x \times \underline{\psi}$$

for irrotational flow

$$\nabla_x \times \underline{v} = -\cancelto{0}{\nabla_x \times \nabla_x \phi} + \nabla_x \times \nabla_x \times \underline{\psi} = 0$$

$$\Rightarrow \underline{\psi} = \underline{0}$$

ϕ is velocity potential for irrotational flow

$$\boxed{\underline{v} = -\nabla_x \phi}$$

in this case $\nabla_x \cdot \underline{v} = 0 \rightarrow -\nabla_x^2 \phi = 0$ Laplace Eqn.

If the flow is irrotational and steady

$$\cancelto{0}{(\nabla_x \times \underline{v})} \times \underline{v} = - \nabla_x H$$

$$\Rightarrow \quad \boxed{\nabla H = 0}$$

H is constant throughout fluid.

Time dependent irrotational flows

Starting from Euler equation

$$\frac{\partial \underline{v}}{\partial t} + \cancelto{0}{(\nabla_x \times \underline{v})} \times \underline{v} = -\frac{1}{2}\nabla_x |\underline{v}|^2 - \frac{1}{\rho_0}\nabla_x p + \underline{b}$$

substituting $\underline{v} = -\nabla_x \phi$

$$\nabla_x \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{v}|^2 + \frac{1}{\rho_0} p - gz \right) = 0$$

which implies that

$$\boxed{\begin{array}{l} \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}|\underline{v}|^2 + \dfrac{p}{\rho_0} - gz = f(t) \\ -\nabla_x^2 \phi = 0 \end{array}}$$

Bernoulli's Thm for irrotational flow

Usefulness hinges on whether irrotational flows are of any real intrest.

$\Rightarrow$ understand evolution of vorticity

Vorticity equation

Define vorticity: $\underline{\omega} = \nabla_x \times \underline{v}$

substitute into Euler eqn.

$$\frac{\partial}{\partial t}\underline{v} + \underline{\omega} \times \underline{v} = -\nabla_x H$$

take the curl

$$\frac{\partial}{\partial t}\underline{\omega} + \nabla_x \times (\underline{\omega} \times \underline{v}) = -\cancelto{0}{\nabla_x \times \nabla H} = 0$$

expand 2^{nd} term $\quad \underline{\omega} = \omega_i \underline{e}_i \quad \underline{v} = v_j e_j$

$$\underline{\omega} \times \underline{v} = \omega_i v_j \, \underline{e}_i \times \underline{e}_k = \underbrace{\omega_i v_j \, \epsilon_{ijk}}_{a_k} \underline{e}_k = a_k \underline{e}_k$$

$$\nabla \times \underline{a} = \epsilon_{kmn} a_{k,n} \underline{e}_m$$

$$= \epsilon_{kmn} \epsilon_{ijk} (\omega_i v_j)_{,n} \underline{e}_m$$

$$= \epsilon_{kmn} \epsilon_{kij} (\omega_i v_j)_{,n} \underline{e}_m$$

$$= (\delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}) (\omega_i v_j)_{,n} \underline{e}_m$$

$$= \delta_{mi} \delta_{nj} (\omega_i v_j)_{,n} \underline{e}_m - \delta_{mj} \delta_{ni} (\omega_i v_j)_{,n} \underline{e}_m$$

$$= (\omega_i v_j)_{,j} \underline{e}_i - (\omega_i v_j)_{,i} \underline{e}_j$$

$$= \omega_{i,j} v_j \underline{e}_i + \omega_i v_{j,j} e_i - \omega_{i,i} v_j e_j - v_{j,i} \omega_i \underline{e}_j$$

$$= (\nabla_x \underline{\omega}) \underline{v} + \cancelto{0}{(\nabla_x \cdot \underline{v})} \underline{\omega} - \cancelto{0}{(\nabla_x \cdot \underline{\omega})} \underline{v} - (\nabla_x \underline{v}) \underline{\omega}$$

So that

$$\frac{\partial}{\partial t}\underline{\omega} + (\nabla_x \underline{\omega})\underline{v} - (\nabla_x \underline{v})\underline{\omega} = 0$$

identifying material derivative

$$\boxed{\dot{\underline{\omega}} - (\nabla_x \underline{v})\underline{\omega} = \underline{0}}$$ Vorticity equation

Show that an initially irrotational fluid remains irrotational!

Simple 2D proof:

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} \qquad \nabla_x \underline{v} = \begin{pmatrix} v_{x,x} & v_{x,y} & 0 \\ v_{y,x} & v_{y,y} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\omega} = \nabla_x \times \underline{v} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$\boxed{(\nabla_x \underline{v})\underline{\omega} = \underline{0} \quad \Rightarrow \quad \dot{\underline{\omega}} = \underline{0}}$$

The vorticity of a fluid element in an ideal fluid is conserved. Vorticity is conserved along streamlines.

In particular, an ideal fluid initially at rest remains irrotational!

Proof in 3D:

Work in material coor.: $\underline{\omega}_m = \underline{\omega}(\underline{\varphi}(\underline{X},t),t)$

Introduce change in variables: $\underline{\omega}_m = \underline{\underline{F}}\,\underline{\xi}$ $\quad \underline{\xi} = \underline{\underline{F}}^{-1}\underline{\omega}_m$

$$\dot{\underline{\omega}} = \frac{\partial}{\partial t}\underline{\omega}_m = \frac{\partial}{\partial t}(\underline{\xi}\,\underline{\underline{F}}) = \dot{\underline{\underline{F}}}\,\underline{\xi} + \underline{\underline{F}}\,\dot{\underline{\xi}}$$

solving for $\dot{\underline{\xi}}$

$$\dot{\underline{\xi}} = \underline{\underline{F}}^{-1}\left[\dot{\underline{\omega}}_m - \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\omega}_m\right]$$

where $\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = (\nabla_x \underline{v})_m$ so that

$$\dot{\underline{\xi}} = \underline{\underline{F}}^{-1}\big[\underbrace{\dot{\underline{\omega}}_m - (\nabla_x \underline{v})_m\,\underline{\omega}_m}_{0 \text{ Vorticity equ}}\big] = 0$$

$$\Rightarrow \quad \dot{\underline{\xi}} = 0 \quad \Rightarrow \quad \underline{\xi}(\underline{X},t) = \underline{\xi}(\underline{X},0)$$

since $\underline{\underline{F}}(\underline{X},0) = \underline{\underline{I}} \Rightarrow \underline{\omega}_m(\underline{X},0) = \underline{\xi}(\underline{X},0)$

$$\begin{aligned}\underline{\omega}_m(\underline{X},t) &= \underline{\underline{F}}(\underline{X},t)\,\underline{\xi}(\underline{X},t)\\ &= \underline{\underline{F}}(\underline{X},t)\,\underline{\xi}(\underline{X},0)\\ &= \underline{\underline{F}}(\underline{X},t)\,\underline{\omega}_m(\underline{X},0)\end{aligned}$$

if $\underline{\omega}_m(\underline{X},0) = 0 \Rightarrow \underline{\omega}_m(\underline{X},t) = 0$!