Isotropic Functions

Functions that are frame indifferent are also called Isotropic Functions

For two reference frames related by a rigid body rotation Q we have:

$$\boxed{\begin{array}{lll}
\phi(\theta) = \phi(\theta) & \phi(\underline{\underline{Q}}\,\underline{v}) = \phi(\underline{v}) & \phi(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \phi(\underline{\underline{S}}) \\
\underline{u}(\theta) = Q\,\underline{u}(\theta) & \underline{u}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\underline{u}(\underline{v}) & \underline{u}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = Q\,\underline{u}(\underline{\underline{S}}) \\
\underline{\underline{\sigma}}(\theta) = \underline{\underline{Q}}\,\underline{\underline{\sigma}}(\theta)\,\underline{\underline{Q}}^T & \underline{\underline{\sigma}}(\underline{\underline{Q}}\,\underline{v}) = \underline{\underline{Q}}\,\underline{\underline{\sigma}}(\underline{v})\,\underline{\underline{Q}}^T & \underline{\underline{\sigma}}(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{\sigma}}(\underline{\underline{S}})\,\underline{\underline{Q}}^T
\end{array}}$$

ϕ = scalar fun. $\quad$ $\underline{u}$ = vector val. fun. $\quad$ $\underline{\underline{\sigma}}$ = tensor val fun.

θ = scalar $\quad$ $\underline{v}$ = vector $\quad$ $\underline{\underline{S}}$ = tensor

Examples: 1) $\phi(\underline{\underline{S}}) = \det(S)$

$$\phi(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}) = \det(\underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \det(\underline{\underline{Q}})\det(\underline{\underline{S}})\det(\underline{\underline{Q}}^T)$$

$$= \det(\underline{\underline{S}}) \quad \checkmark$$

2) $\underline{u}(\underline{v}, \underline{\underline{A}}) = \underline{\underline{A}}\,\underline{v}$

$$\underline{u}(\underline{\underline{Q}}\,\underline{v}, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{v} = \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{v} = Q\,\underline{u}(v, A) \quad \checkmark$$

Representation of isotropic tensor functions

An isotropic function $\underline{\underline{G}}(\underline{\underline{A}}) : \mathcal{V}^2 \to \mathcal{V}^2$ that maps symmetric tensors to symmetric tensors must have the following form

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(I_A)\,\underline{\underline{I}} + \alpha_1(I_A)\,\underline{\underline{A}} + \alpha_2(I_A)\,A^2}$$

Rivlin-Ericksen representation Thm

where α_0, α_1 and α_2 are functions of the set of pricipal invariants of $\underline{\underline{A}}$, $I_A = \{I_1(\underline{\underline{A}}), I_2(\underline{\underline{A}}), I_3(\underline{\underline{A}})\}$

- $\underline{\underline{G}}$ is clearly sym. if $\underline{\underline{A}}$ is sym.
- To see $\underline{\underline{G}}$ is isotropic assume $\alpha_0, \alpha_1\ \alpha_2 = \text{const}$

$$\underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,Q^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underbrace{\underline{\underline{Q}}^T\underline{\underline{Q}}}_{\underline{\underline{I}}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T$$

$$= \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha_2 \underline{\underline{Q}}\,\underline{\underline{A}}^2\,\underline{\underline{Q}}^T$$

$$\underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T = \alpha_0 \underline{\underline{Q}}\,\underline{\underline{Q}}^T + \alpha\, \underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T + \alpha\, \underline{\underline{Q}}\,\underline{\underline{A}}^2\,\underline{\underline{Q}}^T$$

$$\Rightarrow \quad \underline{\underline{G}}(\underline{\underline{Q}}\,\underline{\underline{A}}\,\underline{\underline{Q}}^T) = \underline{\underline{Q}}\,\underline{\underline{G}}(\underline{\underline{A}})\,\underline{\underline{Q}}^T$$

isotropic for constant coefficients.

If coefficeints $\alpha_0, \alpha_1, \alpha_2$ only depend on the invariants of $\underline{\underline{A}}$, then $\underline{\underline{G}}$ remains isotropic.

This is the most general form of a constitutive eqn for an isotropic material.

A second representation can be obtained by eliminating $\underline{\underline{A}}^3$ term using Caley-Hamilton Thm

$$\underline{\underline{A}}^3 - I_1(\underline{\underline{A}})\underline{\underline{A}}^2 + I_2(\underline{\underline{A}})\underline{\underline{A}} - I_3(\underline{\underline{A}})\underline{\underline{I}} = 0$$

multiply by $\underline{\underline{A}}^{-1}$

$$\underline{\underline{A}}^2 - I_1(\underline{\underline{A}})\underline{\underline{A}} + I_2(\underline{\underline{A}})\underline{\underline{I}} - I_3(\underline{\underline{A}})\underline{\underline{A}}^{-1} = 0$$

$$\underline{\underline{A}}^2 = I_1(\underline{\underline{A}})\underline{\underline{A}} - I_2(\underline{\underline{A}})\underline{\underline{I}} + I_3(\underline{\underline{A}})\underline{\underline{A}}^{-1}$$

substituting in to $\underline{\underline{G}}(\underline{\underline{A}})$

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \beta_0(I_A)\underline{\underline{I}} + \beta_1(I_A)\underline{\underline{A}} + \beta_2(I_A)\underline{\underline{A}}^{-1}}$$

$$\beta_0 = \alpha_0 - I_2(\underline{\underline{A}})\alpha_2 \qquad \beta_1 = \alpha_1 - I_1(\underline{\underline{A}})\alpha_2 \qquad \beta_2 = I_3(\underline{\underline{A}})\alpha_2$$

Second representation is used for Hyperelastic materials.

Isotropic Fourth-Order Tensors

If $\underline{\underline{G}}(\underline{\underline{A}})$ is a linear function then it can be written as

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}}$$

where $\mathbb{C}$ is a fourth-order tensor.

If in addition we require:

1) $\mathbb{C}\underline{\underline{A}} \in \mathcal{V}^2$ is symmetric for every symmetric $\underline{\underline{A}} \in \mathcal{V}^2$

2) $\mathbb{C}\,\underline{\underline{W}} = \underline{\underline{0}}$ for every skew-symmetric $\underline{\underline{W}} \in \mathcal{V}^2$

Then there are scalars μ and λ such that

$$\boxed{\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\, \mathrm{tr}(\underline{\underline{A}})\,\underline{\underline{I}} + 2\mu\, \mathrm{sym}(\underline{\underline{A}})} \quad \text{for all } \underline{\underline{A}} \in \mathcal{V}^2$$

This follows from the representation Thm

$$\underline{\underline{G}}(\underline{\underline{A}}) = \alpha_0(\mathcal{I}_A)\,\underline{\underline{I}} + \alpha_1(\mathcal{I}_A)\,\underline{\underline{A}} + \alpha_2(\mathcal{I}_A)\,\underline{\underline{A}}^2$$

where $\mathcal{I}_A = \{ \mathrm{tr}\,\underline{\underline{A}}, \tfrac{1}{2}[(\mathrm{tr}\,\underline{\underline{H}})^2 - \mathrm{tr}(\underline{\underline{H}}^2)], \det \underline{\underline{H}} \}$

since $\underline{\underline{G}}(\underline{\underline{A}})$ is linear in $\underline{\underline{A}}$ the only possibilities are

$\alpha_0(\mathcal{I}_A) = c_0\, \mathrm{tr}\,\underline{\underline{A}} + c_1$, $\alpha_1(\mathcal{I}_A) = c_2$ and $\alpha_2(\mathcal{I}_A) = 0$

where c_0, c_1 and c_2 are scalar constants.

Since $\underline{\underline{G}}(\underline{\underline{0}}) = \underline{\underline{0}} \Rightarrow c_1 = 0$

Hence setting $c_0 = \lambda$ and $c_2 = 2\lambda$

$$\underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}(\underline{\underline{A}}) + 2\mu\,\underline{\underline{A}}$$

since $\underline{\underline{G}}(\underline{\underline{W}}) = 0$ and $\mathrm{tr}(\underline{\underline{W}}) = 0$

$$\Rightarrow \underline{\underline{G}}(\underline{\underline{A}}) = \mathbb{C}\,\underline{\underline{A}} = \lambda\,\mathrm{tr}\,\underline{\underline{A}} + 2\mu\,\mathrm{sym}\,\underline{\underline{A}} \checkmark$$