Linear Elasticity

Initial boundary value problem

PDE: $\rho_0 \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{\hat{P}}}(\underline{\underline{F}}) + \rho_0 \underline{b}_m \qquad \underline{X} \in \Omega \times [0,T]$

BC: $\underline{\varphi} = \underline{g} \qquad \partial\Omega_d$

$\underline{\underline{\hat{P}}}(\underline{\underline{F}})\,\underline{N} = \underline{h} \qquad \partial\Omega_s$

IC: $\underline{\varphi}(\underline{X},0) = \underline{X} \qquad \Omega$

$\dot{\underline{\varphi}}(\underline{X},0) = \underline{V}_0 \qquad \Omega$

Consider a stress-free initial condition

at $t=0$ $\quad \underline{\underline{F}} = \underline{\underline{I}}$ so that $\underline{\underline{\hat{P}}}(\underline{\underline{I}}) = \underline{\underline{\hat{\Sigma}}}(\underline{\underline{I}}) = \underline{\underline{\hat{S}}}_m(\underline{\underline{I}}) = \underline{\underline{0}}$

If all the forcings are small

$|\underline{b}_m| = \mathcal{O}(\varepsilon) \quad |\underline{g} - \underline{X}|_{\partial\Omega_d} = \mathcal{O}(\varepsilon) \quad |\underline{h}| = \mathcal{O}(\varepsilon) \quad |\underline{V}_0| = \mathcal{O}(\varepsilon)$

where $\quad 0 < \varepsilon \ll 1$

In this case, we expect displacements to be small

$$|\underline{u}(\underline{X},t)| = |\underline{\varphi}(\underline{X},t) - \underline{X}| = \mathcal{O}(\varepsilon)$$

Linearized equations

Express forcings as

$\underline{b}_m^\epsilon = \epsilon \, \underline{b}_m \qquad \underline{g}^\epsilon = \underline{X} - \epsilon \underline{g} \qquad \underline{h}^\epsilon = \epsilon \underline{h} \qquad \underline{V}_0^\epsilon = \epsilon \underline{V}_0$

then $\quad \underline{\varphi}^\epsilon = \underline{X} + \epsilon \underline{u} + \mathcal{O} \quad$ where $\quad \underline{u}^\epsilon = \epsilon \underline{u} = \underline{\varphi}^\epsilon - \underline{X}$

and associated $\quad \underline{\underline{F}}^\epsilon = \nabla_X \underline{\varphi}^\epsilon = \underline{\underline{I}} + \epsilon \nabla_X \underline{u}$

substitute into PDE

$$\rho_0 \frac{\partial^2}{\partial t^2}(\underline{X} + \underline{u}^\epsilon) = \nabla_X \cdot [\hat{\underline{\underline{P}}}(\underline{\underline{F}}^\epsilon)] + \rho_0 \underline{b}_m^\epsilon$$

$$\rho_0 \ddot{\underline{u}}^\epsilon = \nabla_X \cdot [\hat{\underline{\underline{P}}}(\underline{\underline{F}}^\epsilon)] + \rho_0 \underline{b}_m^\epsilon$$

Need to deal with $\nabla_X \cdot \hat{\underline{\underline{P}}}(\underline{\underline{F}})$

Introduce the 4th-order tensors

$$A_{ijkl} = \frac{\partial \hat{P}_{ij}}{\partial F_{kl}}(\underline{\underline{I}}) \qquad B_{ijkl} = \frac{\partial \hat{\Sigma}_{ij}}{\partial F_{kl}}(\underline{\underline{I}}) \qquad C_{ijkl} = \frac{\partial \hat{\sigma}_{ij}}{\partial F_{kl}}(\underline{\underline{I}})$$

or in tensor notation the derivatives

$$\mathbb{A}(\underline{\underline{H}}) = \frac{d}{d\alpha} \hat{\underline{\underline{P}}}(\underline{\underline{I}} + \alpha \underline{\underline{H}})\Big|_{\alpha=0} = D\hat{\underline{\underline{P}}}(\underline{\underline{I}})\,\underline{\underline{H}}$$

$$\mathbb{B}(\underline{\underline{H}}) = \frac{d}{d\alpha} \hat{\underline{\underline{\Sigma}}}(\underline{\underline{I}} + \alpha \underline{\underline{H}})\Big|_{\alpha=0} = D\hat{\underline{\underline{\Sigma}}}(\underline{\underline{I}})\,\underline{\underline{H}}$$

$$\mathbb{C}(\underline{\underline{H}}) = \frac{d}{d\alpha} \hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \alpha \underline{\underline{H}})\Big|_{\alpha=0} = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})\,\underline{\underline{H}}$$

Use this to expand stress response

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}^{\epsilon}) = \hat{\underline{\underline{P}}}(\underline{\underline{F}}^{\epsilon})\big|_{\epsilon=0} + \epsilon\,\mathbb{A}(\nabla_x \underline{u}) + \mathcal{O}(\epsilon^2)$$

$$= \cancelto{0}{\hat{\underline{\underline{P}}}(\underline{\underline{I}})} + \epsilon\,\mathbb{A}(\nabla_x \underline{u}) + \mathcal{O}(\epsilon^2)$$

$$= \epsilon\,\mathbb{A}(\nabla_x \underline{u}) + \mathcal{O}(\epsilon^2)$$

substitute into mom. balance with $\underline{u}^{\epsilon} = \epsilon \underline{u}$ & $\underline{b}_m^{\epsilon} = \epsilon \underline{b}_m$

$$\rho_0 \cancel{\epsilon} \ddot{\underline{u}} = \cancel{\epsilon} \nabla_x \cdot [\mathbb{A}(\nabla_x \underline{u})] + \cancel{\epsilon} \rho_0 \underline{b}_m$$

linearized balance of momentum

$$\boxed{\rho_0 \ddot{\underline{u}} = \nabla \cdot [\mathbb{A} \nabla \underline{u}] + \rho_0 \underline{b}}$$ lin. elasto dynamic eqn.

In linearized case, $|\underline{\varphi} - \underline{X}| = \mathcal{O}(\varepsilon)$ and difference between the current and the reference configuration can be neglected.

If accelerations are zero

$$\boxed{\nabla \cdot [\mathbb{A} \nabla \underline{u}] + \rho_0 \underline{b} = \underline{0}}$$ Elasto static eqn.

(Navier-Cauchy eqbm eqn.)

Elasticity Tensor

Introduced three 4-th order tensors:

$\mathbb{A} = D\hat{\underline{\underline{P}}}(\underline{\underline{I}})$, $\mathbb{B} = D\hat{\underline{\underline{\Sigma}}}(\underline{\underline{I}})$ and $\mathbb{C} = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})$

If the reference configuration is stress free

$\Rightarrow \quad \mathbb{A} = \mathbb{B} = \mathbb{C}$

Example: Show $\mathbb{A} = \mathbb{C}$

$$\hat{\underline{\underline{P}}}(\underline{\underline{F}}) = \det(\underline{\underline{F}})\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})\,\underline{\underline{F}}^{-T}$$

differentiating both sides at $\underline{\underline{F}} = \underline{\underline{I}}$

$$\mathbb{A}\,\underline{\underline{H}} = \frac{d}{d\epsilon}\Big[\det(\underline{\underline{I}} + \epsilon\,\underline{\underline{H}})\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \alpha\,\underline{\underline{H}})(\underline{\underline{I}} + \alpha\,\underline{\underline{H}})^{-T}\Big]\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\Big[\det(\underline{\underline{I}} + \epsilon\,\underline{\underline{H}})\Big]\,\cancelto{0}{\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})}\,\underline{\underline{I}}$$

$$+ \cancelto{1}{\det(\underline{\underline{I}})}\,\frac{d}{d\epsilon}\Big[\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\,\underline{\underline{H}})\Big]\Big|_{\epsilon=0}\,\cancelto{\underline{\underline{I}}}{\underline{\underline{I}}^{-T}}$$

$$+ \det(\underline{\underline{I}})\,\cancelto{0}{\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}})}\,\frac{d}{d\epsilon}\Big[(\underline{\underline{I}} + \alpha\,\underline{\underline{H}})^{-T}\Big]$$

where $\det(\underline{\underline{I}}) = 1$ and $\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}}) = \underline{\underline{0}}$ due to stress free initial condition.

$$\Rightarrow \mathbb{A}\,\underline{\underline{H}} = \frac{d}{d\epsilon}\Big[\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\,\underline{\underline{H}})\Big]\Big|_{\epsilon=0} = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}}):\underline{\underline{H}} = \mathbb{C}\,\underline{\underline{H}}$$

so that $\mathbb{A} = \mathbb{C}$ ✓

An elastic solid with stress free IC has a unique elasticity tensor typically denoted $\mathbb{C}$ which can be determined from any stress response function $\hat{\underline{P}}(\underline{\underline{F}})$, $\hat{\underline{\underline{\Sigma}}}(\underline{\underline{F}})$ or $\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})$.

$\Rightarrow$ they have the same linearization at $\underline{\underline{F}} = \underline{\underline{I}}$

Balance of angular momentum

$$\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})^T = \hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})$$

$$[\mathbb{C}\,\underline{\underline{H}}]^T = \left(\frac{d}{d\epsilon}\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \alpha\underline{\underline{H}})\Big|_{\epsilon=0}\right)^T$$

$$= \frac{d}{d\epsilon}\hat{\sigma}(\underline{\underline{I}} + \alpha\underline{\underline{H}})^T\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\hat{\sigma}(\underline{\underline{I}} + \alpha\underline{\underline{H}})\Big|_{\epsilon=0}$$

$$= \mathbb{C}\,\underline{\underline{H}}$$

This implies that $\mathbb{C}$ has left minor symmetry

$$C_{ijkl} = C_{jikl} \quad \text{or} \quad \underline{\underline{A}} : \mathbb{C}\,\underline{\underline{B}} = \mathrm{sym}(\underline{\underline{A}}) : \mathbb{C}\,\mathbb{B}$$

From frame-indifference $\hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{Q}}\,\hat{\underline{\underline{\sigma}}}(\underline{\underline{F}})\underline{\underline{Q}}^T$

and taking $\underline{\underline{F}} = \underline{\underline{I}}$ and $\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}}) = \underline{\underline{0}}$ (stress free IC)

$\Rightarrow \quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}) = \underline{\underline{0}}$

Now we use the fact that an infinitesimal rotation can be written as the matrix exponential, $\exp(\underline{\underline{A}}) = \sum_{j=0}^{\infty} \frac{1}{j!}\underline{\underline{A}}^j = \underline{\underline{I}} + \underline{\underline{A}} + \frac{1}{2}\underline{\underline{A}}^2 + \dots$ of a skew tensor $\underline{\underline{W}} = -\underline{\underline{W}}^T$.

Here we write $\underline{\underline{Q}} = \exp(\epsilon\underline{\underline{W}}) \qquad \epsilon \in \mathbb{R}$

$\Rightarrow \quad \hat{\underline{\underline{\sigma}}}(\underline{\underline{Q}}) = \hat{\underline{\underline{\sigma}}}(\exp(\epsilon\underline{\underline{W}})) = \underline{\underline{0}}$

By definition

$$\mathbb{C}\,\underline{\underline{W}} = D\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}}):\underline{\underline{W}} = \frac{d}{d\epsilon}\hat{\underline{\underline{\sigma}}}(\underline{\underline{I}} + \epsilon\underline{\underline{W}})\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\hat{\underline{\underline{\sigma}}}\Big(\underline{\underline{I}} + \epsilon\underline{\underline{W}} + \frac{\epsilon^2}{2}\underline{\underline{W}}^2 + \frac{\epsilon^3}{6}\underline{\underline{W}}^3 + \dots\Big)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\hat{\underline{\underline{\sigma}}}(\exp(\alpha\underline{\underline{W}}))\Big|_{\epsilon=0} = \underline{\underline{0}}$$

$\Rightarrow \quad \mathbb{C}\,\underline{\underline{W}} = \underline{\underline{0}}$ for any skew tensor

Therefore $\mathbb{C}\underline{\underline{H}} = \mathbb{C}\,\mathrm{sym}(\underline{\underline{H}})$

$\Rightarrow$ $\mathbb{C}$ has right minor symmetry $C_{ijkl} = C_{ijlk}$

In summary:

1) Angular momentum balance: $\boxed{C_{ijkl} = C_{jikl}}$

2) Frame-indifference: $\boxed{C_{ijkl} = C_{ijlk}}$

Elasticity Tensor for Isotropic Solid

Let $\hat{\underline{\underline{\sigma}}}$ be a frame-indifferent Cauchy stress response function for an elastic body with a stress free initial condition. If the body is isotropic the $\mathbb{C}$ is isotropic (Lecture 21) it takes the form

$$\boxed{\mathbb{C}\underline{\underline{H}} = \lambda\,\mathrm{tr}(\underline{\underline{H}})\,\underline{\underline{I}} + 2\mu\,\mathrm{sym}(\underline{\underline{H}})} \qquad \lambda, \mu > 0$$

Linearized Isotropic Elasticity

From the lin. mom. balance

$$\rho_0 \ddot{\underline{u}} = \nabla \cdot (\mathbb{C} \nabla \underline{u}) + \rho_0 \underline{b}$$

where the elasticity tensor is given by

$$\mathbb{C} \nabla \underline{u} = \lambda \, \mathrm{tr}(\nabla \underline{u}) \underline{\underline{I}} + 2\mu \, \mathrm{sym}(\nabla \underline{u})$$

$$\begin{aligned}
\nabla \cdot [\mathbb{C} \nabla u] &= \nabla \cdot [\lambda \, \mathrm{tr}(\nabla \underline{u}) \underline{\underline{I}} + 2\mu \, \mathrm{sym}(\nabla \underline{u})] \\
&= (\lambda u_{k,k} \delta_{ij} + \mu u_{i,j} + \mu u_{j,i})_{,j} \, \underline{e}_i \\
&= (\lambda u_{k,ki} + \mu u_{i,jj} + \mu u_{j,ij}) \underline{e}_i \\
&= \lambda \nabla(\nabla \cdot \underline{u}) + \mu \nabla^2 \underline{u} + \mu \nabla(\nabla \cdot \underline{u}) \\
&= (\lambda + \mu) \nabla(\nabla \cdot \underline{u}) + \mu \nabla^2 \underline{u}
\end{aligned}$$

substituting into lin. mom. bal.

$$\boxed{\rho_0 \ddot{\underline{u}} = \mu \nabla^2 \underline{u} + (\lambda + \mu) \nabla(\nabla \cdot \underline{u}) + \rho_0 \underline{b}}$$ Navier Equation

General linear elastic solid

Stress response function:

$$\hat{\underline{P}}(\underline{\underline{F}}) = \mathbb{C}\,\underline{\underline{e}} \qquad \underline{\underline{e}} = \mathrm{sym}(\nabla \underline{u})$$

Strain energy density

$$W(\underline{\underline{F}}) = \tfrac{1}{2}\,\underline{\underline{e}} : \mathbb{C}\,\underline{\underline{e}}$$

Isotropic model:

$$\mathbb{C}\,\underline{\underline{e}} = \lambda\,\mathrm{tr}(\underline{\underline{e}})\,\underline{\underline{I}} + 2\mu\,\underline{\underline{e}}$$

The St. Venant-Kirchhoff model takes this linear model and extends it to large strain by replacing $\underline{\underline{e}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$ with the Green-Lagrange strain tensor $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$.

Linear: $\hat{\underline{\underline{\sigma}}} = \mathbb{C}\,\underline{\underline{e}} = \lambda\,\mathrm{tr}(\underline{\underline{e}})\,\underline{\underline{I}} + 2\mu\,\underline{\underline{e}}$ $\qquad \underline{\underline{e}} = \nabla \underline{u}$

Non-linear: $\hat{\underline{\underline{\Sigma}}} = \lambda\,\mathrm{tr}(\underline{\underline{E}})\,\underline{\underline{I}} + 2\mu\,\underline{\underline{E}}$ $\qquad \underline{\underline{E}} = \underline{\underline{C}} - \underline{\underline{I}}$

Note latter cannot be written as a 4^{th} order tensor!