Mechanical Equilibrium

Consider a body at rest under the influence of a constant body force, $\rho \underline{b}$, and an external traction, $\underline{h}$. Note : $\underline{b}$ is $\frac{\text{force}}{\text{mass}}$.

Necessary condition for eqbm

A body B is in mech. eqbm if the resultant force and torque (around arbitrary point) vanish for every subset Ω of B. That is

$$\boxed{\left.\begin{aligned} \underline{r}[\Omega] &= r_b[\Omega] + r_s[\partial\Omega] = \int_{\Omega} \rho \underline{b}\, dV + \int_{\partial\Omega} \underline{t}\, dA = 0 \\ \underline{\tau}[\Omega] &= \underline{\tau}_b[\Omega] + \underline{\tau}_s[\partial\Omega] = \int_{\Omega} \underline{x} \times \rho \underline{b}\, dV + \int_{\partial\Omega} \underline{x} \times \underline{t}\, dA = 0 \end{aligned}\right\} \begin{array}{l}\text{for all} \\ \Omega \subseteq B\end{array}}$$

If $\underline{r}[\Omega] = 0$ then $\underline{\tau}[\Omega]$ is independent of $\underline{x}$!

These conditions are intuitive but can also be derived from more general balance laws.

Local Mechanical Equilibrium Equations

If Cauchy stress field $\underline{\underline{\sigma}}$ is continuously differentiable and the density, ρ, and the body force, $\underline{b}$, are continuous, then the equilibrium conditions imply

$$\boxed{\left.\begin{aligned} &\nabla\cdot\underline{\underline{\sigma}}(\underline{x}) + \rho(\underline{x})\, b(\underline{x}) = \underline{0} \\ &\underline{\underline{\sigma}}^T(\underline{x}) = \underline{\underline{\sigma}}(\underline{x}) \end{aligned}\right\} \text{ for all } \underline{x}\in B}$$

or in components

$$\left.\begin{aligned} &\sigma_{ij,j} + \rho\, b_i = 0 \\ &\sigma_{ij} = \sigma_{ji} \end{aligned}\right\} \text{ for all } \underline{x}\in B$$

To establish this we substitute the definition of the Cauchy stress, $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$, into the eqbm conditions

$$\underline{r}[\Omega] = \int_{\partial\Omega} \underline{\underline{\sigma}}\,\underline{n}\, dA + \int_{\Omega} \rho\,\underline{b}\, dV = \underline{0} \qquad \underline{n} = \text{outward normal}$$

using the Tensor Divergence Thm we have

$$\int_\Omega (\nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b})\,dV = \underline{0}$$

by the arbitraryness of Ω the integrand must be zero so that

$$\nabla\cdot\underline{\underline{\sigma}} + \rho\underline{b} = \underline{0}$$

To establish the symmetry of $\underline{\underline{\sigma}}$ we substitute $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$ into the resultant torque

$$\underline{\tau}[\Omega] = \int_{\partial\Omega} \underline{x}\times(\underline{\underline{\sigma}}\,\underline{n})\,dA + \int_\Omega \underline{x}\times\rho\underline{b}\,dV = \underline{0}$$

substituting the previous result $\rho\underline{b} = -\nabla\cdot\underline{\underline{\sigma}}$

$$\int_{\partial\Omega} \underline{x}\times(\underline{\underline{\sigma}}\,\underline{n})\,dA - \int_\Omega \underline{x}\times(\nabla\cdot\underline{\underline{\sigma}})\,dV = \underline{0}$$

to simplify the l.h.s. we define $R_{il} = \epsilon_{ijk}x_j\sigma_{kl}$

which allows us to write $\underline{\underline{R}}\,\underline{n} = \underline{x}\times(\underline{\underline{\sigma}}\,\underline{n})$

$$\int_{\partial\Omega} \underline{\underline{R}}\,\underline{n}\, dA - \int_{\Omega} \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}})\, dV = \underline{0}$$

Applying the Tensor Divergence Thm

$$\int_{\Omega} \nabla \cdot \underline{\underline{R}} - \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}})\, dV = 0$$

by the arbitraryness of Ω we have

$$\nabla \cdot \underline{\underline{R}} - \underline{x} \times (\nabla \cdot \underline{\underline{\sigma}}) = 0 \quad \text{for all } \underline{x} \in B$$

which becomes in components

$$(\epsilon_{ijk} x_j \sigma_{kl})_{,l} - \epsilon_{ijk} x_j \sigma_{kl,l} = 0 \quad \text{for all } \underline{x} \in B$$

using the chain rule

$$\epsilon_{ijk} x_{j,l} \sigma_{kl} + \epsilon_{ijk} x_j \sigma_{kl,l} - \epsilon_{ijk} x_j \sigma_{kl,l} = 0$$

$$\Rightarrow \quad \epsilon_{ijk} x_{j,l} \sigma_{kl} = 0 \qquad \text{with} \quad x_{j,l} = \delta_{jl}$$

$$\epsilon_{ijk} \delta_{jl} \sigma_{kl} = \boxed{\epsilon_{ijk} \sigma_{kj} = 0}$$

If $\epsilon_{ijk} \sigma_{kj} = 0$ then $\epsilon_{ikj} \sigma_{jk} = 0$ because j & k are dummy indices. Hence

$$0 = \epsilon_{ijk} \sigma_{kj} + \epsilon_{ikj} \sigma_{jk} = \epsilon_{ijk} (\sigma_{kj} - \sigma_{jk}) = 0$$

We can always choose i to be distinct from j & k so that $\epsilon_{ijk} \neq 0$ and hence we have

$$\sigma_{kj} = \sigma_{jk}$$