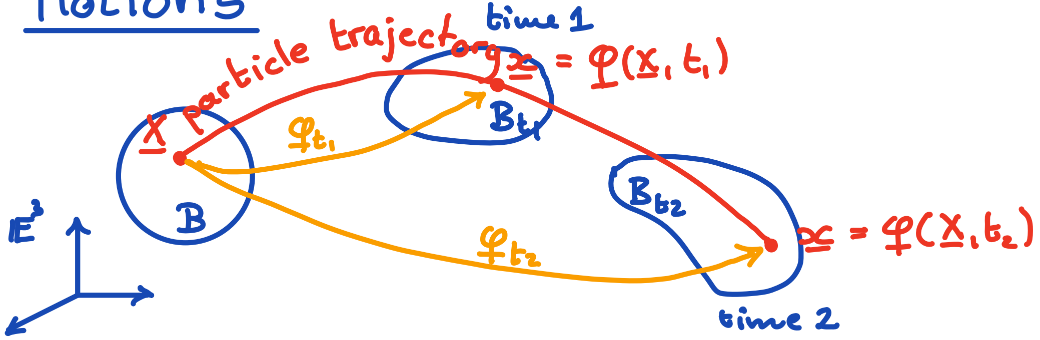
Motions



The continuous deformation of a body over time is called a motion. The motion of a body with ref. configuration $\mathcal{B}$ is described by a continuous map $\varphi : \mathcal{B} \times [0, \infty) \to \mathbb{E}^3$ where for each fixed $t \geq 0$ the function $\underline{\varphi(\cdot, t) = \varphi_t} : \mathcal{B} \to \mathbb{E}^3$ is deformation of $\mathcal{B}$, that maps $\mathcal{B}$ onto a configuration $\mathcal{B}_t = \varphi_t(\mathcal{B})$, called the current or deformed configuration at time t.

We assume that φ_0 is the identity map, so that $\mathcal{B}_0 = \mathcal{B}$.

We assume each φ_t is admissible so that the inverse map $\psi_t = \varphi_t^{-1} : \mathcal{B}_t \to \mathcal{B}$ exists.

$$\boxed{\underline{X} = \underline{\psi}_t(\underline{x}) = \underline{\psi}(\underline{x}, t)}$$

We assume ψ and φ are smooth.

Material and spatial fields

Some fields are naturally defined over the current configuration B_t, for example temperature $T(\underline{x},t)$. Other fields are naturally defined over the reference configuration B, for example grain size $d(\underline{X})$.

However, $\underline{\varphi}$ and $\underline{\psi}$ allow us to represent any function of $\underline{X} \in B$ or of $\underline{x} \in B_t$. To keep track of where a field was originally defined and how it is currently being expressed we introduce following definitions.

Material field is a field expressed in terms of points $\underline{X} \in B$, e.g., $\Omega = \Omega(\underline{X},t)$

Spatial field is a field expressed in terms of points $\underline{x} \in B_t$, e.g., $\Gamma = \Gamma(\underline{x},t)$

To any material field $\Omega(\underline{X},t)$ we associate a spatial field

$$\Omega_s(\underline{x},t) = \Omega(\underline{\psi}(\underline{x},t),t),$$

and call Ω_s the spatial description of Ω.

To any spatial field $\Gamma(\underline{x},t)$ we associate a material field

$$\Gamma_m(\underline{X},t) = \Gamma(\underline{\varphi}(\underline{X},t),t),$$

and call Γ_m the material description of Γ.

Coordinate derivatives

We need to distinguish between derivatives with respect to material and spatial coordinates.

Material coordinates: ∇_X = Grad, Div, Curl, Lap

Spatial coordinates: ∇_x = grad, div, curl, lap

Velocity and acceleration fields

The velocity and acceleration of a material particle labeled $\underline{X}$ in $\mathcal{B}$ at time t due to motion $\underline{\varphi}(\underline{X},t)$

are given by

$$\boxed{\underline{V}(\underline{X},t) = \frac{\partial}{\partial t}\underline{\varphi}(\underline{X},t) = \left.\frac{\partial \underline{x}}{\partial t}\right|_{\underline{X}} \text{ and } \quad \underline{A}(\underline{X},t) = \frac{\partial^2}{\partial t^2}\underline{\varphi}(\underline{X},t) = \left.\frac{\partial^2 \underline{x}}{\partial t^2}\right|_{\underline{X}}}$$

The spatial descriptions of these two fields are

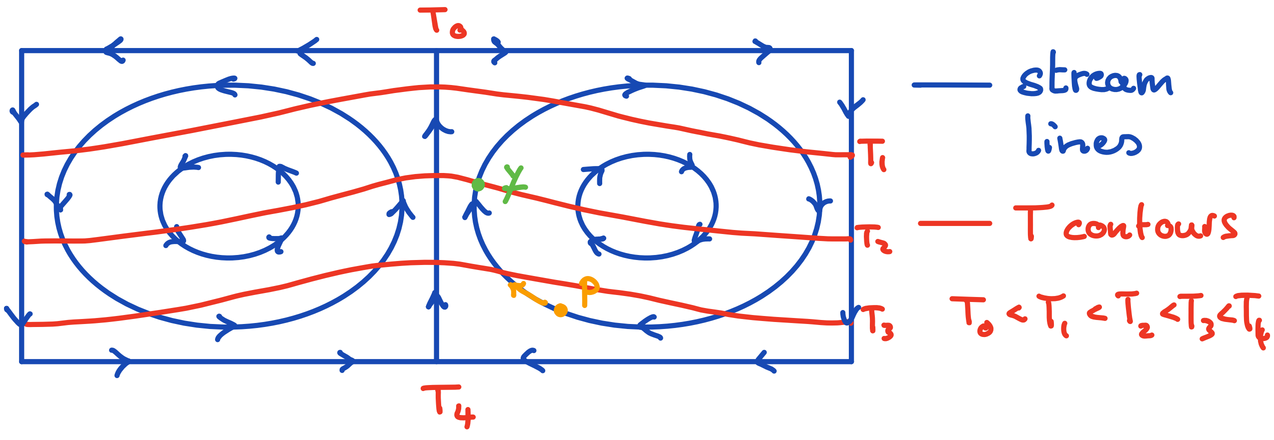
$$\underline{v}(\underline{x},t) = \underline{V}_s(\underline{x},t) = \frac{\partial}{\partial t}\underline{\varphi}(\underline{\psi}(\underline{x},t),t)$$

$$\underline{a}(\underline{x},t) = \underline{A}_s(\underline{x},t) = \frac{\partial^2}{\partial t^2}\underline{\varphi}(\underline{\psi}(\underline{x},t),t)$$

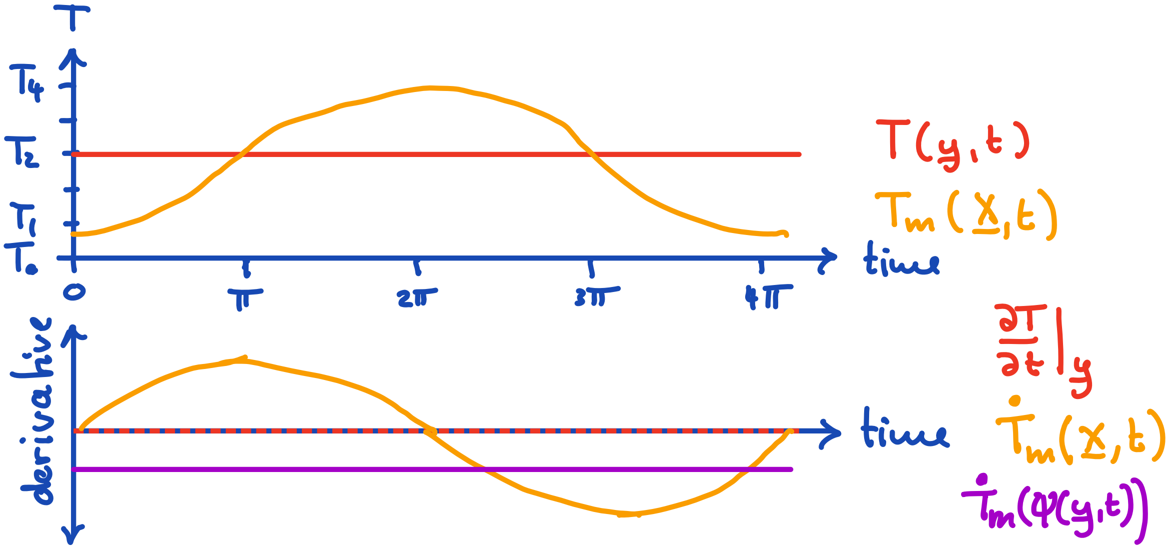
The spatial fields $\underline{v}$ and $\underline{a}$ correspond to the material particle whose current coordinates are $\underline{x}$ at time t.

Note: Below we show that $\boxed{\underline{a} \neq \frac{\partial \underline{v}}{\partial t}}$!

Example: Steady convection



At steady state $T(\underline{x},t) = T(\underline{x})$ (spatial field)



Clearly, $T(y) = T_2$ and $\frac{\partial T}{\partial t}\Big|_y = 0$

Consider a particle P with initial location $\underline{X}$ and current position $\underline{x} = \varphi(\underline{X},t)$, which passes through y once every overturn.

Its temperature is $T(\varphi(\underline{X},t)) = T_m(\underline{X},t)$ is a material field and oscillates periodically. Its derivative the "material derivative" is $\dot{T}_m(\underline{X},t) \neq 0$

Consider the particles passing through y with initial locations $\underline{Y} = \underline{\psi}(y,t)$. What is the change in temperature these particles experience as they pass through y? $\dot{T}_m(\psi(y,t)) = [\dot{T}_m]_s$

Description of field

		material	spatial
Time derivative	material/total	I $\dot{\Omega}(\underline{X},t) = \left.\frac{\partial\Omega}{\partial t}\right\|_{\underline{X}}$	III $\dot{\Gamma}(\underline{x},t) = \frac{\partial\Gamma}{\partial t} + \underline{v}\cdot\nabla\Gamma$ $= \dot{\Omega}(\underline{\psi}(\underline{x},t),t)$
	spatial/Local	$\frac{\partial\Gamma}{\partial t}(\underline{\varphi}(\underline{X},t),t)$	II $\frac{\partial\Gamma}{\partial t} = \left.\frac{\partial}{\partial t}\Gamma(\underline{x},t)\right\|_{\underline{x}}$

Different time derivatives

I) Material time derivative of material field Ω

Derivative of Ω with respect to t, holding $\underset{\sim}{X}$ fixed.

$$\boxed{\dot{\Omega}(\underset{\sim}{X},t) = \frac{D\Omega}{Dt}(\underset{\sim}{X},t) = \frac{\partial \Omega}{\partial t}\bigg|_{\underset{\sim}{X}}}$$

Also called total, substantial or convective derivative.

$\dot{\Omega}$ represents the rate of change of Ω seen by an observer following the path line of a particle.

II) Spatial time derivative of a spatial field $\Gamma(\underset{\sim}{x},t)$

Derivative of Γ with respect to t, holding $\underset{\sim}{x}$ fixed.

$$\frac{\partial \Gamma}{\partial t}(\underset{\sim}{x},t)\bigg|_{\underset{\sim}{x}} = \frac{\partial \Gamma}{\partial t}(\underset{\sim}{x},t)$$

Also referred to as the local time derivative

$\frac{\partial \Gamma}{\partial t}$ represents the rate of change in Γ as seen by an observer at $\underset{\sim}{x}$.

III) Material time derivative of a spatial field

Derivative of scalar spatial field Γ with respect to time t, holding $\underline{X}$ fixed. The material coordinates $\underline{X}$ are fixed while the spatial coordinates change with time $\underline{x} = \underline{\varphi}(\underline{X}, t)$.

$$\boxed{\dot{\Gamma}(\underline{x},t) = \frac{D\Gamma}{Dt}(\underline{x},t) = \frac{\partial}{\partial t}\underbrace{\Gamma(\underline{\varphi}(\underline{X},t),t)}_{\Gamma_m(\underline{X},t)}\Big|_{\underline{X}=\underline{\varphi}(\underline{x},t)}}$$

$\Rightarrow$ two time dependencies, one explicit the other implicit through the motion.

By the chain rule we have for fixed $\underline{X}$

$$\frac{\partial\Gamma}{\partial t}(\underline{\varphi}(\underline{X},t),t) = \frac{\partial\Gamma}{\partial t}(\underline{x},t)\Big|_{\underline{x}=\underline{\varphi}(\underline{X},t)} + \frac{\partial\Gamma}{\partial x_i}(\underline{x},t)\Big|_{\underline{x}=\underline{\varphi}(\underline{X},t)} \frac{\partial}{\partial t}\varphi_i(\underline{X},t)$$

recognizing spatial velocity: $v_i(\underline{x},t)\big|_{\underline{x}=\underline{\varphi}(\underline{X},t)} = \frac{\partial}{\partial t}\varphi_i(\underline{X},t)$

we have the total derivative

$$\frac{\partial\Gamma}{\partial t}(\underline{\varphi}(\underline{X},t),t) = \left[\frac{\partial\Gamma}{\partial t}(\underline{x},t) + \frac{\partial\Gamma}{\partial x_i}(\underline{x},t)\, v_i(\underline{x},t)\right]_{\underline{x}=\underline{\varphi}(\underline{X},t)}$$

this is represented in material coords.

Expressing the result in terms of spatial coords

$$\dot{\Gamma}(\underline{x},t) = \frac{\partial \Gamma}{\partial t}(\underline{x},t) + \frac{\partial \Gamma}{\partial x_i}(\underline{x},t)\, v_i(\underline{x},t)$$

Let $\varphi(\underline{X},t)$ be a motion with spatial velocity field $\underline{v}$ and consider spatial scalar $\phi = \phi(\underline{x},t)$ and vector fields $\underline{\omega} = \underline{\omega}(\underline{x},t)$. Then total time derivatives are given by

$$\boxed{\dot{\phi} = \frac{\partial \phi}{\partial t} + \nabla^x \phi \cdot \underline{v}} \quad \text{and} \quad \boxed{\dot{\underline{\omega}} = \frac{\partial \underline{\omega}}{\partial t} + (\nabla^x \underline{\omega})\underline{v}}$$

The result for $\dot{\underline{\omega}}$ follows by applying the scalar result to ω_i.

This result is important because it allows the computation of $\dot{\phi}$ and $\dot{\underline{\omega}}$ with out knowledge of φ, if the velocity is known!

⇒ in fluid mechanics you never see φ

The spatial acceleration field is defined as

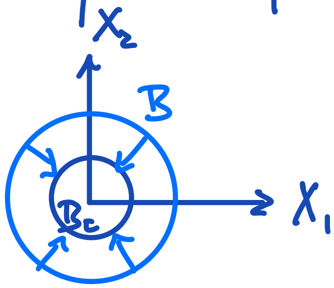
$$\boxed{\underline{a} = \dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla^x \underline{v})\, \underline{v}}$$

Spatial acceleration is a non-linear function of spatial velocity $\Rightarrow$ basic non-linearity in fluid mechanics

Note: $\underline{a} \neq \frac{\partial \underline{v}}{\partial t}$! (only true for mat. fields $\underline{A} = \frac{\partial \underline{V}}{\partial t}$)

Finally, many texts, in particular in fluid mechanics, use the notation $\underline{v} \cdot \nabla^x \underline{\omega}$ instead of $(\nabla^x \underline{\omega})\, \underline{v}$.

Example: Exponential expansion



$$\underline{\varphi}(\underline{X},t) = e^{-\lambda t}\,\underline{X}$$

$$\underline{\psi}(\underline{x},t) = e^{\lambda t}\,\underline{x}$$

Material fields:

$$\underline{V}(\underline{X},t) = \frac{\partial}{\partial t}\underline{\varphi}(\underline{X},t) = -\lambda e^{-\lambda t}\,\underline{X}$$

$$\underline{A}(\underline{X},t) = \frac{\partial}{\partial t}\underline{V}(\underline{X},t) = \lambda^2 e^{-\lambda t}\,\underline{X}$$

Spatial fields:

$$\underline{v}(\underline{x},t) = V_s(\underline{x},t) = V(\underline{\psi}(\underline{x},t),t)$$

$$= -\lambda e^{-\lambda t}\left(e^{\lambda t}\,\underline{x}\right) = -\lambda\,\underline{x}$$

$$\underline{a}(\underline{x},t) = A(\underline{\psi}(\underline{x},t),t) = \lambda^2 e^{-\lambda t}\left(e^{\lambda t}\,\underline{x}\right) = \lambda^2\,\underline{x}$$

Temperature field: $T_m(\underline{X},t) = \alpha\, t\,\|\underline{X}\|$

Material time derivative:

$$\dot{T}_m = \frac{\partial}{\partial t} T_m = \alpha\,\|\underline{X}\|$$

(calculated directly from mat. field)

Spatial Temperature field:

$$T(\underline{x},t) = T_m(\underline{\varphi}(\underline{x},t),t) = \alpha t \, \| e^{\lambda t} \underline{x} \| = \alpha t \|\underline{x}\| e^{\lambda t}$$

Suppose we only know spatial fields $T(\underline{x},t)$ and $\underline{v}(\underline{x},t)$. What is the material derivative $\dot{T}(\underline{x},t)$?

Use $$\dot{T} = \frac{\partial T}{\partial t} + \nabla_x T \cdot \underline{v}$$

$$\frac{\partial T}{\partial t} = \alpha \|\underline{x}\| e^{\lambda t} + \alpha \lambda t \|\underline{x}\| e^{\lambda t}$$

$$\nabla_x T = \alpha t e^{\lambda t} \, \nabla_x (\underline{x}\cdot\underline{x})^{1/2} = \alpha t e^{\lambda t} \, \tfrac{1}{2} (\underline{x}\cdot\underline{x})^{-1/2} \, \nabla_x (\underline{x}\cdot\underline{x})$$

in components: $(x_i x_i)_{,j} = (x_i^2)_{,j} = 2 x_i x_{i,j} = 2 x_i \delta_{ij} = 2 x_j$

$$\Rightarrow \nabla_x (\underline{x}\cdot\underline{x}) = 2\underline{x}$$

$$\nabla_x T = \alpha t e^{\lambda t} \frac{\underline{x}}{\|\underline{x}\|}$$

putting it all together:

$$\dot{T} = \underbrace{\alpha \|\underline{x}\| e^{\lambda t} + \alpha \lambda t \|\underline{x}\| e^{\lambda t}}_{\frac{\partial T}{\partial t}} + \underbrace{\alpha t e^{\lambda t} \frac{\underline{x}}{\|\underline{x}\|}}_{\nabla_x T} \cdot \underbrace{(-\lambda \underline{x})}_{\underline{v}}$$

$$= \alpha \|\underline{x}\| e^{\lambda t} + \alpha \lambda t \|\underline{x}\| e^{\lambda t} - \alpha \lambda t \, \frac{\underline{x}\cdot\underline{x}}{\underbrace{\|\underline{x}\|}} \, e^{\lambda t}$$

$\|\underline{x}\|$

$\dot{T}(\underline{x},t) = \alpha\,\|\underline{x}\|\,e^{\lambda t}$ spatial description of mat. der.

$\dot{T}_m(\underset{\uparrow}{\underline{X}}) = \alpha\,\|\underset{\uparrow}{\underline{X}}\| \longrightarrow \dot{T}_m(\underline{\psi}(\underline{x},t)) = \alpha\,\|\underline{x}\|\,e^{\lambda t} = \dot{T}(\underline{x},t)$

$\underline{X} = \psi(X,t) = e^{\lambda t}\,\underline{x}$