Newtonian Fluids

A fluid is incompressible Newtonian if:

1) Reference mass density uniform: $\rho_0(X) = \rho_0$

2) Fluid is incompressible $\nabla_x \cdot \underline{v} = 0$

3) Cauchy stress field is Newtonian

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + \mathbb{C}\,\nabla_x \underline{v}$$

where $p(x,t)$ is pressure field

and $\mathbb{C}$ is a fourth-order tensor field

with left minor symmetry $(\mathbb{C}\underline{\underline{A}})^T = \mathbb{C}\underline{\underline{A}}$

$\Rightarrow$ ensures the symmetry $\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}} \Rightarrow$ ang. mom.

and trace condition $\operatorname{tr} \mathbb{C}\underline{\underline{A}} = 0$ if $\operatorname{tr} \underline{\underline{A}} = 0$

$\Rightarrow \quad p = \frac{1}{3} \operatorname{tr}(\underline{\underline{\sigma}})$ when $\operatorname{tr}(\nabla_x \underline{v}) = \nabla_x \cdot \underline{v} = 0$

Prop 1 + 2 $\Rightarrow$ $\rho(\underline{x},t) = \rho_0 > 0$

Reactive stress: $\underline{\underline{\sigma}}^r = -p\,\underline{\underline{I}}$

p is multiplier for $\nabla_x \cdot \underline{v} = 0$

Active stress: $\underline{\underline{\sigma}}^a = \mathbb{C}\,\nabla_x \underline{v} = 2\mu \operatorname{sym}(\nabla_x \underline{v})$

by frame indifference

μ = absolute viscosity

In limit $\mu \to 0$ Newtonian fluid reduces to ideal fluid.

Navier - Stokes Equations

Setting $\rho = \rho_0$ and $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu \,\mathrm{sym}(\nabla_x \underline{v})$

we obtain lin. mom. balance

$$\rho_0 \dot{\underline{v}} = \nabla_x \cdot \left(-p\underline{\underline{I}} + 2\mu \,\mathrm{sym}(\nabla_x \underline{v})\right) + \rho_0 \underline{b}$$

from mat. deriv. $\dot{\underline{v}} = \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v}$

assuming μ = constant we have

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu \nabla_x \cdot \nabla_x \underline{v} + \mu \nabla_x \cdot (\nabla_x \underline{v})^T$$

$$\nabla_x \cdot \nabla_x \underline{v} = v_{i,jj}\, \underline{e}_i = \nabla_x^2 \underline{v}$$

$$\nabla_x \cdot (\nabla_x \underline{v})^T = v_{j,ij}\, \underline{e}_i = v_{j,ji}\, \underline{e}_i = \nabla_x (\cancel{\nabla_x \cdot \underline{v}}^{\,0})$$

$$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} = -\nabla_x p + \mu \nabla_x^2 \underline{v}$$

so that

$$\boxed{\begin{gathered} \rho_0 \left[\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v}\right] = \mu \nabla_x^2 \underline{v} - \nabla_x p + \rho_0 b \\ \nabla_x \cdot \underline{v} = 0 \end{gathered}}$$

Frame-Indifference of Newtonian fluid model

We already showed the indifference of constraint.

$\Rightarrow$ focus on active stress

$$\underline{\underline{\sigma}}^a = \mathbb{C}\, \nabla_x \underline{v} = 2\mu\, \mathrm{sym}(\nabla_x \underline{v}) = 2\mu \underline{\underline{d}}$$

Check left minor symmetry of $\mathbb{C}$

$$(\mathbb{C}\, \nabla_x \underline{v})^T = (2\mu\, \underline{\underline{d}})^T = 2\mu\, \underline{\underline{d}}^T = 2\mu \underline{\underline{d}} = \mathbb{C}\, \nabla_{\underline{x}} \underline{v} \quad \checkmark$$

Check trace condition

$$\mathrm{tr}(\mathbb{C}\, \nabla_x \underline{v}) = 2\mu\, \mathrm{tr}(\underline{\underline{d}}) = 0 \quad \text{if} \quad \mathrm{tr}(\underline{\underline{d}}) = 0$$

Now assume a superposed rigid motion

$$\underline{x}^* = \underline{\underline{Q}}(t)\, \underline{x} + \underline{c}(t)$$

where $\quad \underline{\underline{\sigma}}^{a*}(\underline{x}^*, t) = \underline{\underline{\sigma}}^{a*} \quad$ and $\quad \underline{\underline{d}}^* = \underline{\underline{d}}^*(\underline{x}^*, t)$

$$\underline{\underline{\sigma}}^{a*} = 2\mu\, \underline{\underline{d}}^* = 2\mu\, \underbrace{\underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^*}_{\underline{\underline{d}}^*} = \underline{\underline{Q}}(2\mu \underline{\underline{d}})\underline{\underline{Q}}^T = \underline{\underline{Q}}\, \underline{\underline{d}}\, \underline{\underline{Q}}^T \quad \checkmark$$

see also discussion in Lecture 20

Mechanical energy considerations

Stress power of Newtonian fluid is

$$\underline{\underline{\sigma}} : \underline{\underline{d}} = (-pI + 2\mu \underline{\underline{d}}) : \underline{\underline{d}} = -p \underbrace{\underline{\underline{I}} : \underline{\underline{d}}}_{\nabla_x \cdot \underline{v} = 0} + 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

$$= 2\mu\, \underline{\underline{d}} : \underline{\underline{d}}$$

From reduced Clausius-Duhem inequality

$$\rho \dot{\psi} \leq 2\mu \underbrace{\underline{\underline{d}} : \underline{\underline{d}}}_{\geq 0}$$

$\Rightarrow$ only if $\mu > 0$ energy is dissipated during the flow $\dot{\psi} < 0$

Kinetic Energy of Fluid Motion

Dissipation of Kinetic energy in Ideal and Newtonian fluids.

First some useful results:

1) Integration by parts in fixed domain Ω with "no slip" boundaries $\underline{v} = \underline{0}$ on $\partial\Omega$.

$$\boxed{\int_\Omega (\nabla_x^2 \underline{v}) \cdot \underline{v}\, dV_x = -\int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v})\, dV_x}$$

To see this consider $(v_{i,j} v_i)_{,j} = v_{i,jj} v_i + v_{i,j} v_{i,j}$

$$(\nabla_x^2 \underline{v}) \cdot v = v_{i,jj} v_i = (v_{i,j} v_i)_{,j} - v_{i,j} v_{i,j}$$

$$= \nabla \cdot ((\nabla_x \underline{v})^T \underline{v}) - (\nabla_x \underline{v}) : (\nabla_x \underline{v})$$

substituting into integral and applying div-thm

$$\int_\Omega (\nabla_x^2 \underline{v}) \cdot \underline{v} \, dV_x = \int_{\partial\Omega} (\nabla_x v)^T \cancelto{0}{\underline{v}} \cdot \underline{n} \, dA_x - \int_\Omega (\nabla_x \underline{v}) : (\nabla_x \underline{v}) \, dV_x$$

2) Poincaré Inequality

$$\| \underline{u} \|_\Omega \le \lambda \| \nabla_x u \|_\Omega \quad \text{for} \quad \underline{u} = 0 \quad \partial\Omega \quad \lambda > 0$$

using standard inner product

$$\boxed{\int_\Omega |\underline{u}|^2 \, dV_x \le \lambda \int_\Omega \nabla_x \underline{u} : \nabla_x \underline{u} \, dV_x}$$

Notice λ has units of L^2 and scales with area of Ω.

Kinetic Energy of Newtonian & Ideal fluids

Consider a fixed domain Ω with $\underline{v} = 0$ on $\partial\Omega$

and a conservative body force $b = -\nabla_x \Phi$.

The kinetic energy is given by

$$K(t) = \int_{\Omega} \frac{1}{2} \rho_0 |\underline{v}|^2 \, dV_x \quad \text{and} \quad K(0) = K_0$$

I) Newtonian fluid

$$\boxed{K(t) \leq e^{-2\mu t/\lambda \rho_0} K_0}$$

The kinetic energy of a Newtonian fluid dissipates to zero exponentially fast.

II) Ideal fluid

$$\boxed{K(t) = K_0}$$

The kinetic energy of ideal fluid is constant.

By def. of K we have

$$\frac{d}{dt}K(t) = \int_\Omega \frac{1}{2}\rho_0 \frac{d}{dt}|\underline{v}|^2\, dV_x = \int_\Omega \rho_0\, \dot{\underline{v}}\cdot\underline{v}\, dV_x$$

from Navier-Stokes Eqns: $\rho_0\dot{\underline{v}} = \mu\nabla_x^2\underline{v} - \nabla\underline{\Psi}$

$$\frac{d}{dt}K(t) = \int_\Omega\left(\mu\nabla_x^2\underline{v} - \nabla\underline{\Psi}\right)\cdot\underline{v}\, dV_x$$

show $\int_\Omega \nabla_x\psi\cdot\underline{v}\, dV_x = 0$

$$\nabla_x\cdot(\varphi\underline{v}) = \nabla_x\varphi\cdot\underline{v} + \cancel{(\nabla_x\cdot\underline{v})}^{\,0}\varphi = \nabla_x\varphi\cdot\underline{v}$$

substitute and use Div-Thm

$$\frac{d}{dt}K(t) = \int_\Omega \mu\left(\nabla_x^2\underline{v}\right)\cdot\underline{v}\, dV_x - \int_{\partial\Omega}\psi\,\cancel{\underline{v}}^{\,0}\cdot\underline{n}\, dA_x$$

using integration by parts

$$\frac{d}{dt}K(t) = -\mu\int_\Omega(\nabla_x\underline{v}):(\nabla_x\underline{v})\, dV_x$$

for Ideal fluid $\mu = 0 \Rightarrow K(t) = K_0$

for Newtonian fluid apply Pointcare inequality

$$\frac{d}{dt}K(t) \le -\frac{\mu}{\lambda}\int_\Omega|\underline{v}|^2\, dV_x = -\frac{2\mu}{\lambda\rho_0}K(t)$$

so that we have

$$\boxed{\frac{d}{dt}K(t) \le -\frac{2\mu}{\lambda\rho_0}K(t)}$$

where λ depends on area of the domain.

Solve by seperation of parts

$$\frac{dK}{K} \leq -\frac{2\mu}{\rho_0 \lambda}\, dt = -\alpha\, dt$$

$$\ln K \leq -\alpha t + c_0$$

$$K \leq c_1 e^{-\alpha t}$$

Initial condition $\quad K(0) \leq c_1 = K_0$

$$\Rightarrow \quad K(t) \leq K_0\, e^{-\frac{2\mu}{\lambda \rho_0} t} \quad \checkmark$$

In absence of fluid motion on the boundary fluid motion decays exponentially.

The rate of decay depends

$$\boxed{\nu = \frac{\mu}{\rho_0}} \quad \text{kinematic viscosity}$$

Scaling Navier Stokes Equations

$$\rho_0 \frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\, \underline{v} = \mu \nabla_x^2 \underline{v} - \nabla_x p + \rho g$$

reduced pressure:

$$-\nabla_x p + \rho g = -\nabla_x p - \rho g \hat{z} = -\nabla(p + \rho g z) = -\nabla \pi$$

we have

$$\rho_0 \left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\, \underline{v} \right) - \mu \nabla_x^2 \underline{v} = -\nabla_x \pi$$

Non-dimensionalize with generic quantities to define standard dimensionless parameters.

- Dependent variables: $\underline{v}$, π
- Independent variables: $\underline{x}$, t
- Parametess: $\rho \left[\frac{M}{L^3}\right]$ $\mu \left[\frac{M}{LT}\right] \rightarrow \nu = \frac{\mu}{\rho} \left[\frac{L^2}{T}\right]$

+ Geometry, BC, IC

Use parameters to scale the variables:

$$\underline{v}' = \frac{\underline{v}}{v_c} \qquad \pi' = \frac{\pi}{\pi_c} \qquad \underline{x}' = \frac{\underline{x}}{x_c} \qquad t' = \frac{t}{t_c}$$

substitute into governing equations

$$\frac{\rho_0 v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho v_c^2}{x_c}(\nabla_x' \underline{v}')\underline{v}' - \frac{\mu v_c}{x_c^2}\nabla_x'^2 \underline{v}' = -\frac{\pi_c}{x_c}\nabla_x' \pi'$$

Option 1: Scale to accumulation term

$$\frac{\partial \underline{v}'}{\partial t'} + \underbrace{\frac{v_c t_c}{x_c}}_{\Pi_1}(\nabla_x' \underline{v}')\underline{v}' - \underbrace{\frac{\nu t_c}{x_c^2}}_{\Pi_2}\nabla_x'^2 \underline{v}' = -\underbrace{\frac{\pi_c t_c}{x_c \rho_0 v_c}}_{\Pi_3}\nabla_x' \pi'$$

where $\nu = \frac{\mu}{\rho}$ "momentum diffusivity"

Three dimensionless groups $\Rightarrow$ define time scale

$\Pi_1 = \frac{v_c t_c}{x_c} = 1 \Rightarrow$ advective scale $t_c = t_A = \frac{x_c}{v_c}$

$\Pi_2 = \frac{\nu t_c}{x_c^2} = 1 \Rightarrow$ diffusive scale $t_c = t_D = \frac{x_c^2}{\nu}$

Use Π_3 to define pressure scale

$\Pi_3 = \frac{\pi_c t_c}{x_c \rho_0 v_c} = 1 \Rightarrow \pi_c = \frac{x_c \rho_0 v_c}{t_c}$

Choose a diffusive time scale $t_c = \frac{x_c^2}{\nu}$

$$\frac{\partial \underline{v}}{\partial t} + \frac{v_c x_c}{\nu}(\nabla_x' \underline{v}')\underline{v}' - \nabla_x'^2 \underline{v} = -\nabla_x' \pi'$$

$\Rightarrow$ one remaining dim. less group

$\boxed{Pe_m = \frac{t_D}{t_A} = \frac{v_c x_c}{\nu} = Re}$ Reynolds number

Hence we have

$$\frac{\partial \underline{v}'}{\partial t'} + Re\,(\nabla'_x \underline{v}')\underline{v}' - \nabla'^2_x \underline{v} = -\nabla'_x \pi'$$

Advective momentum transport vanishes as $Re \to 0$

For viscous flow of glacier:

$\rho_0 = 10^3 \frac{kg}{m^3}$ $\quad v_c = 100 \frac{m}{yr} \sim 10^{-6} \frac{m}{s}$

$\mu = 10^{14}$ Pas $\quad x_c = 10^2 m$ (thickness)

$$Re = \frac{v_c x_c \rho_0}{\mu_c} = \frac{10^{-6+2+3}}{10^{14}} = 10^{-1-14} = 10^{-15} \ll 1$$

$\Rightarrow$ advective momentum transport is negligible

Momentum balance simplifies

$$\frac{\partial \underline{v}'}{\partial t} - \nabla'^2_x \underline{v} = -\nabla'_x \pi'$$

But is it worth resolving diffusive timescale?

$$t_D = \frac{x_c^2 \rho_0}{\mu} = 10^{4+3-14} s = 10^{-7} s$$

This is very short compared to 100 years of glacier response. Not worth resolving transients.

Can't eliminate transient term because we scaled to it $\Rightarrow$ scale to diffusion term.

Stokes Equation

Scaling to mom. diffusion

$$\frac{\rho_0 v_c}{t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{\rho_0 v_c^2}{x_c}(\nabla_x' \underline{v}')\underline{v}' - \frac{\mu v_c}{x_c^2}\nabla_x'^2 \underline{v}' = -\frac{\pi_c}{x_c}\nabla_x' \pi'$$

divide by $\mu v_c / x_c^2$

$$\frac{x_c^2}{\nu t_c}\frac{\partial \underline{v}'}{\partial t'} + \frac{v_c x_c}{\nu}(\nabla_x' \underline{v}')\underline{v}' - \nabla_x'^2 \underline{v} = -\underbrace{\frac{\pi_c x_c}{\mu v_c}}_{1}\nabla_x' \pi'$$

choose $t_c = t_A = \frac{x_c}{v_c}$ $\qquad 1 \Rightarrow \pi_c = \frac{\mu v_c}{x_c}$

$$Re\left(\frac{\partial \underline{v}}{\partial t} + (\nabla_x \underline{v})\underline{v}\right) - \nabla_x'^2 \underline{v} = -\nabla_x' \pi'$$

In the limit $Re \ll 1$ we obtain

$$\boxed{\begin{aligned} \nabla_x'^2 \underline{v}' &= \nabla_x' \pi' \\ \nabla_x' : \underline{v}' &= 0 \end{aligned}}$$

Stokes equations dimension less

Redimensionalize : $\underline{v}' = \frac{\underline{v}}{v_c}$ $\quad \pi' = \frac{\pi}{\frac{\mu v_c}{x_c}}$ $\quad x' = \frac{x}{x_c}$

$$\frac{x_c^2}{v_c}\nabla_x^2 \underline{v} = \frac{x_c^2}{\mu v_c}\nabla_x \pi$$

$$\boxed{\begin{aligned} \mu \nabla_x^2 \underline{v} &= \nabla_x \pi \\ \nabla_x : \underline{v} &= 0 \end{aligned}}$$

Dimensional Stokes equation

Properties of the Stokes Equation

1) Linearity

   Construct solutions by linear superposition

2) Instanteneity

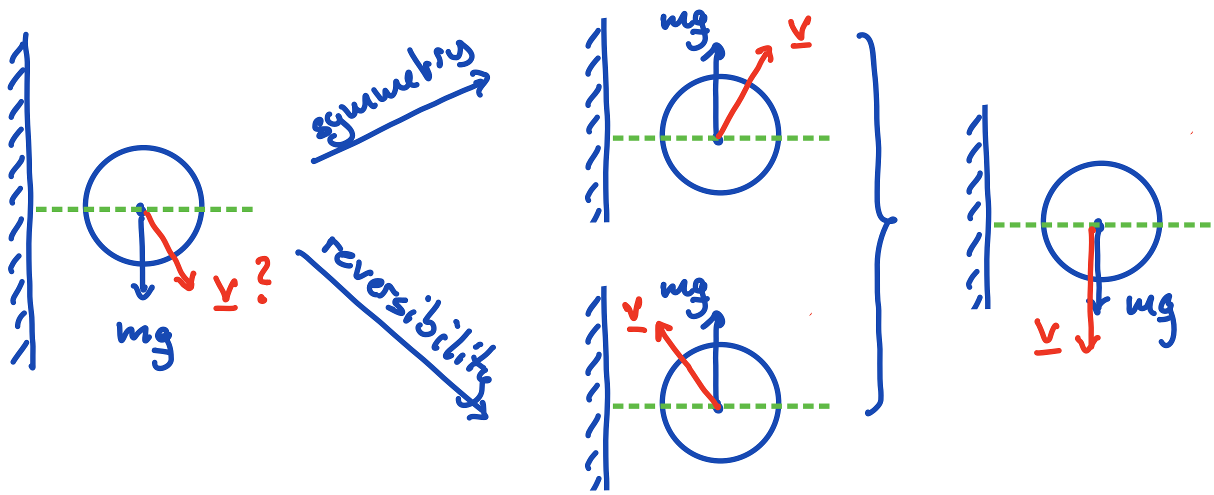
   No time dependence other than due to time varying boundary conditions

3) Reversibility

   If the body force and the velocity on boundary are reversed so is the velocity everywhere.

These tell us a lot about possible solutions.

Example: Sphere falling next to a wall



Energy dissipation in Stokes flow

Dissipation rate : $\mathcal{D} = \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV = 2\mu \int_\Omega \underline{\underline{d}} : \underline{\underline{d}} \, dV$

In a Stokes flow there is no inertia and hence all energy dissipated must be supplied as external power.

$$\mathcal{D} = \int_\Omega \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x = \int_\Omega \underline{\underline{\sigma}} : \nabla_x \underline{v} \, dV_x$$

from PS 4: $\nabla \cdot (\underline{\underline{\sigma}} \, \underline{v}) = \underline{\underline{\sigma}}^T : \nabla \underline{v} + \underline{v} \cdot \nabla \cdot \underline{\underline{\sigma}}^T$

$$\mathcal{D} = \int_\Omega \nabla_x \cdot (\underline{\underline{\sigma}} \, \underline{v}) \, dV_x - \int \underline{v} \cdot \nabla_x \cdot \underline{\underline{\sigma}} \, dV$$

for Stokes $\nabla_x \cdot \underline{\underline{\sigma}} = -\rho \underline{b}$ (was absorbed in $\nabla \pi$!)

use div. theorem

$$= \int_{\partial\Omega} \underline{\underline{\sigma}} \, \underline{v} \cdot \underline{n} \, dA_x + \int_\Omega \underline{v} \cdot \rho \underline{b} \, dV \qquad \underline{\underline{\sigma}} \, \underline{v} \cdot \underline{n} = \underline{v} \cdot \underbrace{\underline{\underline{\sigma}}^T \underline{n}}_{\underline{t}}$$

$$= \int_{\partial\Omega} \underline{v} \cdot \underline{t} \, dA_x + \int_\Omega \underline{v} \cdot \rho \underline{b} \, dV = \mathcal{P}$$

$\boxed{\mathcal{D} = \mathcal{P}}$ $\qquad$ $\mathcal{P}$ is the external power

Helmholtz minimum dissipation Theorem

For a given domain and boundary conditions the rate of dissipation in a Stokes flow is less or equal to any other incompressible flow.

Stokes flow : $\underline{v}$ $\underline{\underline{\sigma}}$ $\qquad \nabla \cdot \underline{v} = 0$

Other flow : $\underline{v}'$ $\underline{\underline{\sigma}}'$ $\underline{\underline{d}}'$ $\qquad \nabla \cdot \underline{v} = 0$

Dissipation :

$$\mathcal{D} = 2\mu \int \underline{\underline{d}} : \underline{\underline{d}} \, dV$$

$$\mathcal{D}' = 2\mu \int \underline{\underline{d}}' : \underline{\underline{d}}' \, dV = 2\mu \int \underline{\underline{d}} : \underline{\underline{d}} + (\underline{\underline{d}} - \underline{\underline{d}}') : (\underline{\underline{d}} - \underline{\underline{d}}') \, dV$$

$$d'd' - d'd - dd' + dd$$