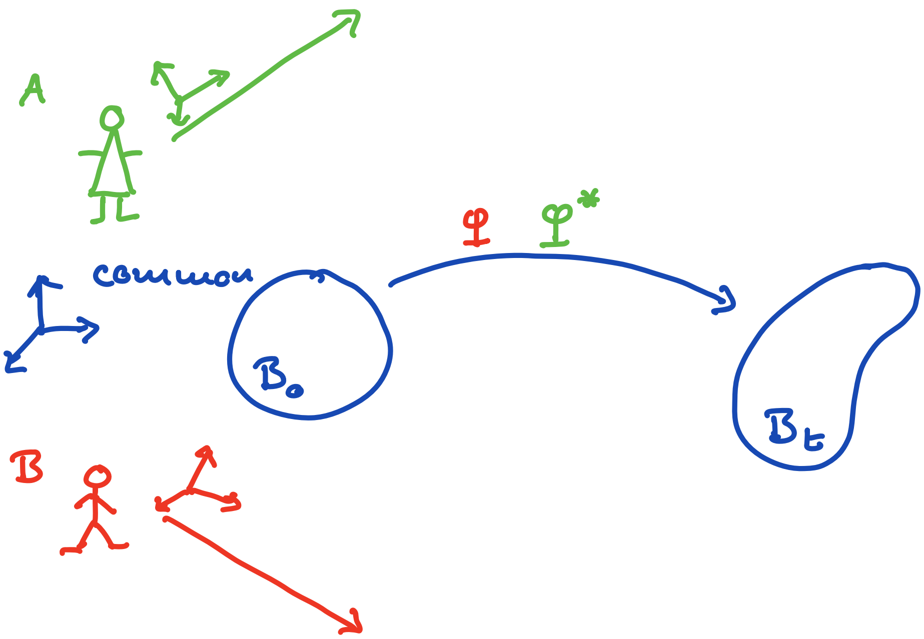
Frame - Indifference



Two observers in diff. ref. frames $\{\underline{e}_i^*\}$ and $\{\underline{e}_i\}$ must be related by a rigid motion in a common frame.

$$\boxed{\varphi^*(\underline{X},t) = \underline{\underline{Q}}(t)\,\varphi(\underline{X},t) + \underline{c}(t)}$$ Euclidian transformation

where $\underline{\underline{Q}}(t)$ and $\underline{c}(t)$ are the rotation and translation of one observer relative to the other.

Note: We assume all observers use same clock.

The effect of superposed rigid motion on the kinematic material fields is

$$\underline{\underline{F}}^* = \underline{\underline{Q}}\,\underline{\underline{F}} \qquad \underline{\underline{R}}^* = \underline{\underline{Q}}\,\underline{\underline{R}} \qquad \underline{\underline{U}}^* = \underline{\underline{U}}$$

$$\underline{\underline{V}}^* = \underline{\underline{Q}}\,\underline{\underline{V}}\,\underline{\underline{Q}}^T \qquad \underline{\underline{C}}^* = \underline{\underline{C}}$$

note that $\underline{X}^* = \underline{X}$, ie the material frame is independent of the observer. The spatial coordinates, in contrast are different:

$$\underline{x} = \varphi(\underline{X}, t) \quad \text{and} \quad \underline{x}^* = \varphi^*(\underline{X}, t)$$

The relations between the spatial kinematic fields are:

$$\underline{\underline{\ell}}^* = \nabla_{x^*} \underline{v}^* = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$$

$$\underline{\underline{d}}^* = \mathrm{sym}(\underline{\underline{\ell}}^*) = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$$

where $\dot{\underline{\underline{Q}}}(t) = \frac{\partial \underline{\underline{Q}}}{\partial t}$ because $\underline{\underline{Q}}$ is constant in space.

For example: $\varphi^* = \underline{\underline{Q}}(t)\,\varphi(\underline{X},t) + \underline{c}(t)$

$$\varphi = \underline{\underline{F}}(\underline{X},t)\,\underline{X} + \underline{a}(t)$$

$$\Rightarrow \underline{\underline{F}}^* = \nabla\varphi^* = \underline{\underline{Q}}(t)\,\underline{\underline{F}}(\underline{X},t)$$

From def. of $\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$ we have

$$\underline{\underline{C}}^* = \underline{\underline{F}}^{*T}\underline{\underline{F}}^* = (\underline{\underline{Q}}\,\underline{\underline{F}})^T(\underline{\underline{Q}}\,\underline{\underline{F}}) = \underline{\underline{F}}^T\underline{\underline{Q}}^T\underline{\underline{Q}}\,\underline{\underline{F}} =$$

$$= \underline{\underline{F}}^T\underline{\underline{F}} = \underline{\underline{C}}$$

From lecture 15: $\underline{\underline{\ell}} = \nabla_{\underline{x}}\underline{v}\big|_{\underline{x}=\varphi(\underline{X},t)} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

$$\underline{\underline{\ell}}^* = \nabla_{\underline{x}^*}\underline{v}^*\big|_{\underline{x}^*=\varphi^*(\underline{X},t)} = \dot{\underline{\underline{F}}}^*\,\underline{\underline{F}}^{*-1}$$

$$\dot{\underline{\underline{F}}}^* = \frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{F}}) = \dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}$$

$$\underline{\underline{F}}^{*-1} = (\underline{\underline{Q}}\,\underline{\underline{F}})^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$\Rightarrow \underline{\underline{\ell}}^* = \dot{\underline{\underline{F}}}^*\,\underline{\underline{F}}^{*-1} = (\dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}})\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T$$

$$= \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T \quad \text{where } \underline{\underline{\ell}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$$

$$\underline{\underline{\ell}}^* = \underline{\underline{Q}}\,\underline{\underline{\ell}}\,\underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$$

we can show $\underline{\underline{d}}^* = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$ by noticing that $\dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T$ is skew symmetric.

Axiom of Frame Indifference

The fields ϕ, $\underline{w}$, $\underline{\underline{S}}$ are called frame-indifferent if for all superposed motions $\underline{x}^* = \underline{\underline{Q}}\,\underline{x} + \underline{c}$ we have

$$\boxed{\begin{aligned} \phi^*(\underline{x}^*, t) &= \phi(\underline{x}, t) \\ \underline{w}^*(\underline{x}^*, t) &= \underline{\underline{Q}}\,\underline{w}(\underline{x}, t) \\ \underline{\underline{S}}^*(\underline{x}^*, t) &= \underline{\underline{Q}}\,\underline{\underline{S}}(\underline{x}, t)\,\underline{\underline{Q}}^T \end{aligned}}$$

Some physical quantities associated with the body are inherent to the body and independent of the observer.

Note, not all fields are frame indifferent!
Two observers in relative motion will disagree on the velocity and acceleration fields.

Note, frame indifferent tensors are also called objective.

Example: Is the stress tensor $\underline{\underline{\sigma}} = 2\mu \nabla_x \underline{v}$ objective?

Using result from above

$$\underline{\underline{\sigma}}^* = 2\mu \left(\underline{\underline{Q}} \nabla_x \underline{v} \, \underline{\underline{Q}}^T + \dot{\underline{\underline{Q}}} \, \underline{\underline{Q}}^T \right)$$

$$= \underline{\underline{Q}} (2\mu \nabla_x \underline{v}) \underline{\underline{Q}}^T + 2\mu \dot{\underline{\underline{Q}}} \underline{\underline{Q}}^T$$

$$= \underline{\underline{Q}} \, \underline{\underline{\sigma}} \, \underline{\underline{Q}}^T + 2\mu \dot{\underline{\underline{Q}}} \underline{\underline{Q}}^T \neq \underline{\underline{Q}} \, \underline{\underline{\sigma}} \, \underline{\underline{Q}}^T$$

$\Rightarrow$ not objective/frame indifferent.

On the other hand $\underline{\underline{\sigma}} = 2\mu \underline{\underline{d}}$ is objective

because $\underline{\underline{\sigma}}^* = 2\mu \underline{\underline{Q}} \, \underline{\underline{d}} \, \underline{\underline{Q}}^T = \underline{\underline{Q}} \, 2\mu \underline{\underline{d}} \, \underline{\underline{Q}}^T = \underline{\underline{Q}} \, \underline{\underline{\sigma}} \, \underline{\underline{Q}}^T$!

$\Rightarrow$ constitutive laws are base on the symmetric parts of strain or rate of strain tensors.

Galilean transformations

Consider velocity under change of observer.

$$\underline{x}^* = \varphi^*(\underline{X},t) = \underline{\underline{Q}}(t)\,\varphi(\underline{X},t) + \underline{c}(t)$$

also $\quad \underline{x} = \underline{\underline{Q}}^T(t)\left(\underline{x}^* - \underline{c}(t)\right)$

The velocity fields are

$$\underline{v}(\underline{x},t) = \underline{V}(\underline{X},t) = \dot{\varphi}(\underline{X},t)$$

$$\underline{v}^*(\underline{x}^*,t) = \underline{V}^*(\underline{X},t) = \dot{\varphi}^*(\underline{X},t)$$

$$= \dot{\underline{\underline{Q}}}(t)\,\varphi(\underline{X},t) + \underline{\underline{Q}}\,\dot{\varphi}(\underline{X},t) + \dot{\underline{c}}(t)$$

$$\underline{v}^* = \dot{\underline{\underline{Q}}}\,\underline{x} + \underline{\underline{Q}}\,\underline{v} + \dot{\underline{c}}$$

$$\underline{v}^* = \underline{\underline{Q}}\,\underline{v} + \dot{\underline{\underline{Q}}}\,\underline{x} + \dot{\underline{c}}$$

$\Rightarrow$ $\underline{v}$ is only objective if $\dot{\underline{\underline{Q}}} = \dot{\underline{c}} = 0$

if $\underline{\underline{Q}}$ and $\underline{c}$ are constants

substituting $\quad \underline{x} = \underline{\underline{Q}}^T(\underline{x}^* - \underline{c})$

$$\underline{v}^* = \underline{\underline{Q}}\,\underline{v} + \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T(\underline{x}^* - \underline{c}) + \dot{\underline{c}}$$

$$\boxed{\underline{v}^* = \underline{\underline{Q}}\,\underline{v} + \dot{\underline{c}} + \underline{\underline{\Omega}}(\underline{x}^* - \underline{c})}$$

where $\underline{\underline{\Omega}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T = -\underline{\underline{\Omega}}^T$ spin of $*$ frame relative to other acceleration

$$\dot{\underline{a}}^* = \dot{\underline{v}}^* = \underline{\underline{Q}}\,\dot{\underline{v}} + \ddot{\underline{c}} + \dot{\underline{\underline{Q}}}\,\underline{v} + \dot{\underline{\underline{\Omega}}}(\underline{x}^* - \underline{c}) + \underline{\underline{\Omega}}(\dot{\underline{x}}^* - \dot{\underline{c}})$$

subst. $\underline{v} = \dot{\underline{\underline{Q}}}^T(\underline{x}^* - \underline{c}) + \underline{\underline{Q}}^T(\underline{v}^* - \dot{\underline{c}})$

$$\underline{\underline{\Omega}}^2 = -\dot{\underline{\underline{Q}}}\,\dot{\underline{\underline{Q}}}^T$$

$$\boxed{\dot{\underline{a}}^* = \underline{\underline{Q}}\,\underline{a} + \ddot{\underline{c}} + (\dot{\underline{\underline{\Omega}}} - \underline{\underline{\Omega}})(\underline{x}^* - \underline{c}) + 2\Omega(\underline{v}^* - \dot{\underline{c}})}$$

$\Rightarrow$ $\underline{a}$ is not ob'ective

3 fictive accelerations

1. Euler acceleration: $\dot{\underline{\underline{\Omega}}}(\underline{x}^* - \underline{c})$
2. Centrifugal acc.: $-\underline{\underline{\Omega}}^2(\underline{x}^* - \underline{c})$
3. Coriolis acc.: $2\underline{\underline{\Omega}}(\underline{v}^* - \dot{\underline{c}})$

Acceleration is only objective under a Galilean transformation: $\ddot{\underline{c}} = 0$, $\dot{\underline{\underline{Q}}} = 0$

$$\boxed{\underline{x}^* = \underline{\underline{Q}}\,\underline{x} + \underbrace{\underline{c}_0 + v_0 t}_{c(t)}}$$

If two observers move relative to each other with constant velocity they measure the same acceleration

Frame Indifferent functions

The spatial mass density, $\rho(\underline{x},t)$, temperature $\theta(\underline{x},t)$, energy and entropy densities $u(\underline{x},t)$ and $s(\underline{x},t)$, the Cauchy stress $\underline{\underline{\sigma}}(\underline{x},t)$ and heatflow $\underline{q}(\underline{x},t)$ are frame-indifferent fields.

Hence constitutive functions such as

$$\phi(\underline{x},t) = \hat{\phi}(\rho(\underline{x},t), \theta(\underline{x},t), \underline{\underline{S}}(\underline{x},t))$$

$$\underline{q}(\underline{x},t) = \hat{\underline{q}}(\rho, \theta, \underline{\underline{S}})$$

$$\underline{\underline{\sigma}}(\underline{x},t) = \hat{\underline{\underline{\sigma}}}(\rho, \theta, \underline{\underline{S}})$$

This requires that the constitutive functions $\hat{\phi}$, $\hat{\underline{q}}$ and $\hat{\underline{\underline{\sigma}}}$ are independent of frame of the input variables

$$\hat{\phi}(\rho^*, \theta^*, \underline{\underline{S}}^*) = \hat{\phi}(\rho, \theta, \underline{\underline{S}}^*)$$

since $\rho^* = \rho$ $\theta^* = \theta^*$ and $\underline{\underline{S}}^* = \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T$

$$\hat{\phi}(\rho, \theta, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) = \hat{\phi}(\rho, \theta, \underline{\underline{S}})$$

since $\hat{\underline{q}}(\rho^*, \theta^*, \underline{\underline{S}}^*) = \underline{\underline{Q}}\,\underline{q}(\rho, \theta, \underline{\underline{S}})$

and $\quad \hat{\underline{\underline{\sigma}}}(p^*, \theta^*, \underline{\underline{S}}^*) = \underline{\underline{Q}}\, \hat{\underline{\underline{\sigma}}}(p, \theta, \underline{\underline{S}})\, \underline{\underline{Q}}^T$

we have the following constraints

$$\boxed{\begin{aligned}
\hat{\phi}(p, \theta, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) &= \hat{\phi}(p, \theta, \underline{\underline{S}}) \\
\hat{\underline{q}}(p, \theta, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) &= \underline{\underline{Q}}\, \hat{\underline{q}}(p, \theta, \underline{\underline{S}}) \\
\hat{\underline{\underline{\sigma}}}(p, \theta, \underline{\underline{Q}}\,\underline{\underline{S}}\,\underline{\underline{Q}}^T) &= \underline{\underline{Q}}\, \underline{\underline{\sigma}}(p, \theta, \underline{\underline{S}})\, \underline{\underline{Q}}^T
\end{aligned}}$$

$\Rightarrow$ Constitutive functions must be written in terms of invariants of $\underline{\underline{S}}$ that are independent of $\underline{\underline{Q}}$!