Power-law creep and non-Newtonian viscosity

Newtonian fluid : $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + C\,\nabla\underline{v}$ <u>linear</u>

Note: Constitutive law is linear but the

kin. momentum balance is still non-linear.

In Earth science most important non-Newtonian rheology is <u>power-law creep</u>:

$$\boxed{\dot{\varepsilon} = A\,\sigma^n}$$

$\dot{\varepsilon}$ = strain rate

σ = stress

n = stress exponent

A = pre-factor

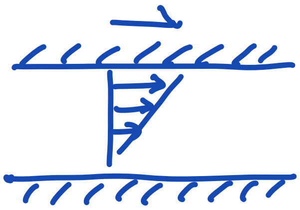
Note this is a scalar relation we need proper tensor form.

This behavior is common in polycrystalline solids close to their melting point.

We follow exposition in "Rheology of the Earth"

The general tensor form can be established from experiments.

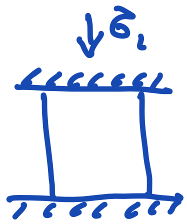
<u>Simple shear</u>



$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{d}} = \underline{\underline{\dot{\varepsilon}}} = \begin{bmatrix} 0 & \dot{\varepsilon}_s & 0 \\ \dot{\varepsilon}_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Uniaxial compression</u>



$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{d}} = \underline{\underline{\dot{\varepsilon}}} = \begin{bmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{approx. ?}$$

Suppose shear experiments lead to relation

$$\dot{\varepsilon}_s = A\, \sigma_s^{\,n}$$

A is function of P, T & material parameters but stress exponent n is constant.

How do we extend experimental result to

a general state of stress?

What is tensor form of power-law creep?

1) Experiments are not affected by pressure

$\Rightarrow$ use deviatoric stress & strain rate

2) Frame indifferent $\rightarrow$ use invariants

Invariants from Lecture

$$I_1(\underline{\underline{S}}) = tr(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 = S_{11} + S_{22} + S_{33}$$

$$I_2(\underline{\underline{S}}) = \frac{1}{2}\left(tr(\underline{\underline{S}})^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$-I_2(\underline{\underline{S}}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} =$$

$$\sigma_{11}\sigma_{22} - \sigma_{12}^2 + \sigma_{11}\sigma_{33} - \sigma_{13}^2 + \sigma_{22}\sigma_{33} - \sigma_{23}^2$$

$$I_2(\underline{\underline{S}}) = -\left(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}\right) + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2$$

$$I_3(\underline{\underline{S}}) = det(\underline{\underline{S}}) = \lambda_1\lambda_2\lambda_3$$

Not sure why I_3 is not considered further

likely $I_3(\dot{\underline{\underline{\varepsilon}}}) = \nabla_x \cdot \underline{v} = 0$

Invariants of deviatoric tensors $\underline{\underline{\sigma}}'$ $\underline{\underline{\dot{\varepsilon}}}'$

$J_1(\underline{\underline{\sigma}}) = J_1(\underline{\underline{\dot{\varepsilon}}}) = 0$ by definition

$J_2(\underline{\underline{\sigma}}) = \frac{1}{2}\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'$ $\qquad J_2(\underline{\underline{\dot{\varepsilon}}}) = \frac{1}{2}\,\underline{\underline{\dot{\varepsilon}}}' : \underline{\underline{\dot{\varepsilon}}}'$

To see this

$J_2(\underline{\underline{\sigma}}) = \sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{13}'^2$ because $\sigma_{11}' = \sigma_{22}' = \sigma_{33}' = 0$

$\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}' = \sigma_{12}'^2 + \sigma_{21}'^2 + \sigma_{13}'^2 + \sigma_{31}'^2 + \sigma_{23}'^2 + \sigma_{32}'^2 = 2\,J_2(\underline{\underline{\sigma}})$ by symmetry

If material is incompressible $J_2(\underline{\underline{\dot{\varepsilon}}}) = I_2(\underline{\underline{\dot{\varepsilon}}}) = \frac{1}{2}\,\underline{\underline{\dot{\varepsilon}}} : \underline{\underline{\dot{\varepsilon}}}$

We can define effective stress and strain rate

$$\sigma_E' = \sqrt{\tfrac{1}{2}\,\underline{\underline{\sigma}} : \underline{\underline{\sigma}}} \qquad \dot{\varepsilon}_E = \sqrt{\tfrac{1}{2}\,\underline{\underline{\dot{\varepsilon}}} : \underline{\underline{\dot{\varepsilon}}}}$$

and rewrite the power-law as

$$\dot{\varepsilon}_E = A\,\sigma_E'^{\,n}$$

in terms of invariants and hence objective.

To extend this to tensorial form we <u>assume</u>

$$\underline{\underline{\dot{\varepsilon}}} = \lambda(\sigma_E)\,\underline{\underline{\sigma}}'$$

which is reasonable for isotropic materials.

$$\dot{\varepsilon}_E = \sqrt{\tfrac{1}{2}\underline{\underline{\dot{\varepsilon}}} : \underline{\underline{\dot{\varepsilon}}}} = \sqrt{\tfrac{1}{2}\lambda^2\,\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'} = \lambda\sqrt{\tfrac{1}{2}\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'}$$

$$\Rightarrow \qquad \dot{\varepsilon}_E = \lambda\,\sigma_E'$$

combining $\dot{\varepsilon}_E = A\,\sigma_E'^{\,n}$ and $\dot{\varepsilon}_E = \lambda\,\sigma_E'$

we have $\lambda\,\sigma_E' = A\,\sigma_E'^{\,n} \Rightarrow \lambda = A\,\sigma_E'^{\,n-1}$

Tensor form of power-law creep

$$\boxed{\underline{\underline{\dot{\varepsilon}}} = A\,\sigma_E'^{\,(n-1)}\,\underline{\underline{\sigma}}'}$$

Compare to Representation Thm

$$\underline{\underline{\dot{\varepsilon}}}(\underline{\underline{\sigma}}') = \alpha_0(I_\sigma)\,\underline{\underline{I}} + \alpha_1(I_\sigma)\,\underline{\underline{\sigma}}' + \alpha_2(I_\sigma)\,\underline{\underline{\sigma}}'^{\,2}$$

we see that $\alpha_0 = \alpha_2 = 0$ $\qquad \alpha_1 = A\,\sigma_E^{(n-1)} = \alpha_1(I_\sigma(\sigma'))$ ✓

$\Rightarrow$ frame indifferent

Example: Simple shear

$$\underline{\underline{\sigma}} = \begin{pmatrix} 0 & \sigma_s & 0 \\ \sigma_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{\underline{\sigma}}' \qquad \underline{\underline{\dot{\varepsilon}}} = \begin{pmatrix} 0 & \dot{\varepsilon}_s & 0 \\ \dot{\varepsilon}_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{\underline{\dot{\varepsilon}}}'$$

$$\sigma_E' = \sqrt{\tfrac{1}{2}\underline{\underline{\sigma}}' : \underline{\underline{\sigma}}'} = \sqrt{\tfrac{1}{2}(\sigma_s^2 + \sigma_s^2)} = \sigma_s \quad \text{similarly} \quad \dot{\varepsilon}_E = \dot{\varepsilon}_s$$

substitute $\underline{\underline{\dot{\varepsilon}}} = A\,\sigma_E'^{\,n-1}\,\underline{\underline{\sigma}} \Rightarrow \dot{\varepsilon}_s = A\,\sigma_E'^{\,n-1}\sigma_s$

only non-zero components $\dot{\varepsilon}_{12} = \dot{\varepsilon}_{21} = \dot{\varepsilon}_s \quad \sigma_{12} = \sigma_{21} = \sigma_s$

$\Rightarrow \dot{\varepsilon}_s = A\,\sigma_s^{\,n-1}\sigma_s = A\,\sigma_s^n$ ✓

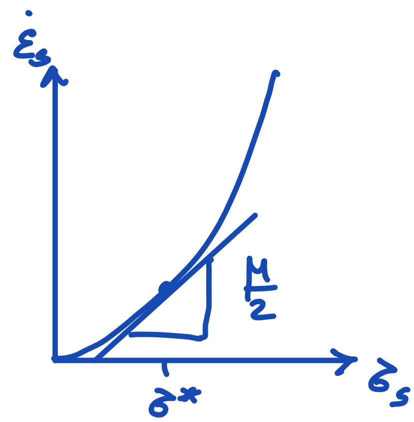
Effective viscosity of power-law creep

Standard Newtonian Fluid

$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\mu\,\underline{\underline{d}} \quad (\underline{\underline{d}} = \underline{\underline{\dot{\varepsilon}}})$

$\underline{\underline{\sigma}}' = 2\mu\,\underline{\underline{\dot{\varepsilon}}}'$

$\Rightarrow \boxed{\mu = \dfrac{\underline{\underline{\sigma}}'}{2\,\underline{\underline{\dot{\varepsilon}}}'} = \dfrac{\sigma_s}{2\,\dot{\varepsilon}_s}}$



What are A and n ?

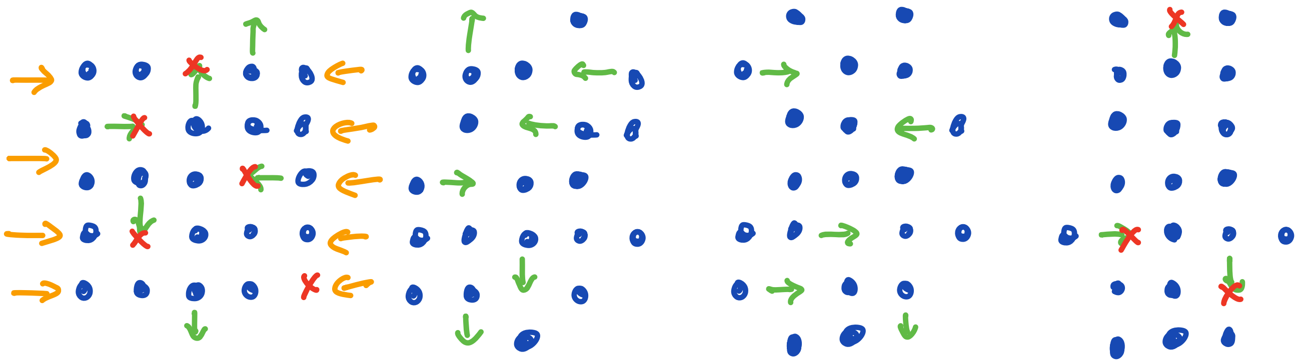
$\Rightarrow$ depends on deformation mechanism

Atomic basis of creep deformation

Solid deforms by motion of lattice defects.

I) Diffusion creep

Due to diffusion of lattice vacancies



Vacancies migrate to relieve the horizontal stress on the lattice. This process reduces the number of vacancies and hence the energy of the lattice.

Diffusion creep leads to Newtonian behaviour

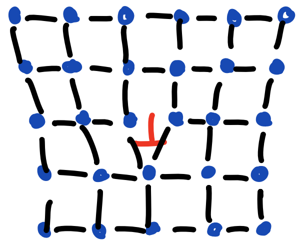
$$\underline{\underline{\sigma}}' = 2\mu \dot{\underline{\underline{\varepsilon}}} \qquad \text{with} \qquad \mu = \frac{RTd^2}{24 V_a D_0} \exp\left(\frac{E_a + pV_a}{RT}\right)$$

strongly grain size and temperature dependent.

II) Dislocation Creep

Imperfections in crystal lattice

· Edge dislocation



Skrew dislocation