Rate of Deformation Tensors

→ role of deformation gradient in rates

Velocity gradients

Spatial velocity gradient

$$\underline{\underline{l}} = \nabla_x \underline{v} \qquad\qquad l_{ij} = \frac{\partial v_i}{\partial x_j}$$

Material velocity gradient

$$\underline{\underline{F}} = \nabla_X \underline{\varphi} \qquad F_{iJ} = \varphi_{i,J} \quad \text{and} \quad \underline{V} = \dot{\underline{\varphi}} \qquad V_i = \varphi_{i,t}$$

$$\Rightarrow \dot{\underline{\underline{F}}} = \frac{\partial}{\partial t}(\nabla_X \underline{\varphi}) = \nabla_X\left(\frac{\partial}{\partial t}\underline{\varphi}\right) = \nabla_X \underline{V} \qquad \dot{F}_{iJ} = V_{i,J}$$

Note analogy

$$\underline{\varphi}(\underline{X} + \underline{\Delta X}, t) \approx \underline{\varphi}(\underline{X}, t) + \underline{\underline{F}}(\underline{X}, t)\, \underline{\Delta X} \qquad \underline{\underline{F}} = \nabla \underline{\varphi}$$

taking material derivative $\quad \dot{\varphi} = V$

$$\underline{V}(\underline{X} + \Delta X, t) \approx \underline{V}(\underline{X}, t) + \dot{\underline{\underline{F}}}(\underline{X}, t)\, \underline{\Delta X} \qquad \dot{\underline{\underline{F}}} = \nabla \underline{V}$$

Note: $\underline{V}(\underline{X}, t) = \underline{v}(\underline{\varphi}(\underline{X}, t), t)$

⇒ same vector field just expressed in $\underline{X}$ or $\underline{x}$

$$\nabla_X V \neq \nabla_x \underline{v}\big|_{\underline{x} = \varphi(\underline{X}, t)}$$

because derivatives are in different directions

To relate $\nabla_x \underline{v}$ and $\nabla_X \underline{V}$ use $\underline{V}(\underline{X},t) = \underline{v}(\varphi(\underline{X},t),t)$

$$\dot{F}_{iJ} = \frac{\partial}{\partial X_J} V_i = \frac{\partial}{\partial X_J} v_i(\varphi(\underline{X},t),t)$$

where $\frac{\partial}{\partial X_J} = \frac{\partial}{\partial x_k}\frac{\partial x_k}{\partial X_J} = \frac{\partial}{\partial x_k}\varphi_{k,J} = \frac{\partial}{\partial x_k}F_{kJ}$

substituting

$$\dot{F}_{iJ} = \frac{\partial}{\partial X_J} v_i(\varphi(\underline{X},t),t) = \frac{\partial}{\partial x_k} v_i(\underline{x},t)\, F_{kJ}$$

$$= v_{i,k}\, F_{kJ}$$

$\Rightarrow$ $\dot{\underline{\underline{F}}} = \nabla_x \underline{v}\, \underline{\underline{F}} = \underline{\underline{\ell}}\, \underline{\underline{F}}$ or $\nabla_X \underline{V} = \nabla_x \underline{v}\, \underline{\underline{F}}$

also $\boxed{\underline{\underline{\ell}} = \nabla_x \underline{v} = \dot{\underline{\underline{F}}}\, \underline{\underline{F}}^{-1}}$ $\quad \ell_{ij} = \dot{F}_{iK} F^{-1}_{Kj}$

To understand $\underline{\underline{\ell}}$ we need to decompose it

similar to $\underline{\underline{F}} = \nabla\varphi$ and $H = \nabla \underline{u}$

finite strain: $\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$

infinitesimal strain: $\underline{\underline{H}} = \mathrm{sym}(\underline{\underline{H}}) + \mathrm{skew}(\underline{\underline{H}})$

Decomposition of $\underline{\underline{l}}$

Split into sym. and skew

$$\boxed{\begin{aligned} \underline{\underline{l}}(\underline{x},t) &= \underline{\underline{d}}(\underline{x},t) + \underline{\underline{w}}(\underline{x},t) \\ \underline{\underline{d}} &= \tfrac{1}{2}\left(\nabla_x \underline{v} + \nabla_x \underline{v}^T\right) \\ \underline{\underline{w}} &= \tfrac{1}{2}\left(\nabla_x \underline{v} - \nabla_x \underline{v}^T\right) \end{aligned}}$$

rate of strain tensor

spin tensor

Interpretation of $\underline{\underline{d}}$ and $\underline{\underline{l}}$

$$\underline{v}(\underline{x}+\Delta\underline{x},t) \approx \underline{v}(\underline{x},t) + \nabla_x \underline{v}\,\Delta\underline{x} \qquad \nabla_x \underline{v} = \underline{\underline{l}} = \underline{\underline{d}} + \underline{\underline{w}}$$

$$\approx \underline{v}(\underline{x},t) + \underline{\underline{d}}\,\Delta\underline{x} + \underline{\underline{w}}\,\Delta\underline{x}$$

because $\underline{\underline{w}}$ is skew $\to$ axial vector $\underline{\omega} = \mathrm{vec}(\underline{\underline{w}})$

so that $\quad \underline{\underline{w}}\,\Delta\underline{x} = \underline{\omega} \times \Delta\underline{x}$

$$\Rightarrow \underline{v}(\underline{x}+\Delta\underline{x},t) \approx \underline{v}(\underline{x},t) + \underline{\underline{d}}\,\Delta\underline{x} + \underline{\omega} \times \Delta\underline{x}$$

$\Rightarrow$ $\underline{\underline{d}}$ is rate of change in shape (strech rate)

$\Rightarrow$ $\underline{\underline{w}}$ is rate of change in orientation (spin)

where $\underline{\omega}$ is the angular velocity.

$\Rightarrow$ vorticity: $\nabla_x \times \underline{v} = 2\underline{\omega}$ $\;\to 2$ the spin

Relation of $\underline{\underline{\ell}}$ to $\dot{\underline{\underline{U}}}$ and $\dot{\underline{\underline{R}}}$

$$\underline{\underline{\ell}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$$

Need material derivatives of $\underline{\underline{F}}^{-1}$ and $\underline{\underline{F}}^{-T}$.

$$\dot{\underline{\underline{I}}} = \frac{d}{dt}\left(\underline{\underline{F}}^{-1}\underline{\underline{F}}\right) = \dot{\underline{\underline{F}}}^{-1}\underline{\underline{F}} + \underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}} = \underline{\underline{0}}$$

$$\dot{\underline{\underline{F}}}^{-1} = \underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{\ell}}$$

$$\dot{\underline{\underline{F}}}^{-T} = \left(\dot{\underline{\underline{F}}}^{-1}\right)^{T} = \left(\underline{\underline{F}}^{-1}\underline{\underline{\ell}}\right)^{T} = \underline{\underline{\ell}}\,\underline{\underline{F}}^{-T}$$

$$\Rightarrow \quad \boxed{\dot{\underline{\underline{F}}}^{-1} = \underline{\underline{F}}^{-1}\underline{\underline{\ell}} \qquad \dot{\underline{\underline{F}}}^{-T} = \underline{\underline{\ell}}\,\underline{\underline{F}}^{-T}}$$

$$\underline{\underline{\ell}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1} = \frac{d}{dt}\left(\underline{\underline{R}}\,\underline{\underline{U}}\right)\underline{\underline{F}}^{-1} = \left(\dot{\underline{\underline{R}}}\,\underline{\underline{U}} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\right)\underline{\underline{F}}^{-1}$$

$$= \left(\dot{\underline{\underline{R}}}\,\underline{\underline{U}} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\right)\left(\underline{\underline{R}}\,\underline{\underline{U}}\right)^{-1} = \left(\dot{\underline{\underline{R}}}\,\underline{\underline{U}} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\right)\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T}$$

$$= \dot{\underline{\underline{R}}}\,\underline{\underline{U}}\,\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T} = \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^{T} + \underline{\underline{R}}\,\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1}\underline{\underline{R}}^{T}$$

show that $\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^{T} = -\left(\dot{\underline{\underline{R}}}\,\underline{\underline{R}}\right)^{T}$ $\Rightarrow$ show $\left(\frac{d}{dt}\left(\underline{\underline{R}}\,\underline{\underline{R}}^{T}\right) = 0\right)$

$$\Rightarrow \quad \boxed{\begin{aligned} \underline{\underline{d}} &= \underline{\underline{R}}\ \mathrm{sym}\left(\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1}\right)\underline{\underline{R}}^{T} \\ \underline{\underline{w}} &= \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^{T} + \underline{\underline{R}}\ \mathrm{skew}\left(\dot{\underline{\underline{U}}}\,\underline{\underline{U}}^{-1}\right)\underline{\underline{R}}^{T} \end{aligned}}$$

Shows that $\underline{\underline{d}}$ is not a pure rate of strain

and $\underline{\underline{w}}$ is not a pure rate of rotation

<u>Interpretation of $\underline{\underline{d}}$ & $\underline{\underline{w}}$</u>

By analogy with infinitesimal strain and rotation tensors

Diagonal components of $\underline{\underline{w}}$ quantify the instantaneous rate of streching of line segments at $\underline{x}$ in B_t, which are aligned with the coordinate axes.

Similarly, the off-diagonal components of $\underline{\underline{d}}$ quantify the instantaneous rate of shearing between coord. directions.

The tensor $\underline{\underline{w}}$ quantifies the instantaneous rate of rigid rotation at $\underline{x}$ in B_t. The axial vector $\underline{w} = \text{vec}(\underline{\underline{w}})$ is related to the vorticity $\underline{\omega} = \nabla_x \times \underline{v} = 2\underline{w}$. Hence, vorticity measures the rate of rotation or spin at $\underline{x}$. The vorticity measures twice the angular velocity at $\underline{x}$.