Review of Vectors

Def: Vector is a quantity with a magnitude & direction.

$$\underline{v} = |\underline{v}|\,\hat{\underline{v}}$$

$|\underline{v}|$ = magnitude $(|\underline{v}| \geq 0)$

$\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$ direction $(|\hat{\underline{v}}| = 1)$ unit vector

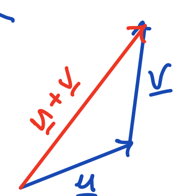
Examples: force, velocities, displacements, ...

Q: Is it possible to have vector without direction?

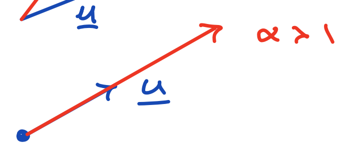
Def: Vector space, $\mathcal{V}$, is a collection of objects that is closed under addition and scalar multiplication.

$\underline{u} \in \mathcal{V}$ $\underline{v} \in \mathcal{V}$ $\alpha \in \mathbb{R}$

1) $\underline{u} + \underline{v} \in \mathcal{V}$



2) $\alpha\,\underline{u} \in \mathcal{V}$

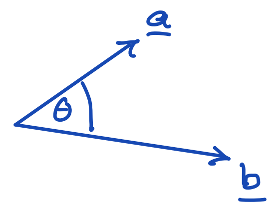


Q1: Do vectors in $\mathbb{R}^3$ form vector space?

Q2: Do vectors in $\mathbb{R}^+$ form vector space?

Scalar product: $\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta} \qquad \theta \in [0, \pi]$$



$\underline{a} \cdot \underline{b} = 0 \qquad \underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{a} = |\underline{a}|^2$

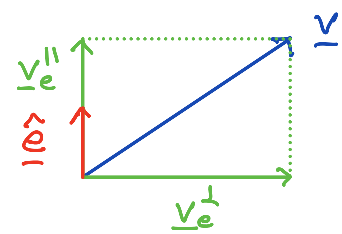
$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ commutative

Projection: $\hat{\underline{e}}$ unit vector & $\underline{v} \in \mathcal{V}$

$\underline{v} = \underline{v}_e^{\parallel} + \underline{v}_e^{\perp}$

$\underline{v}_e^{\parallel} = \underline{v} \cdot \underline{e}$

$\underline{v}_e^{\perp} = \underline{v} - \underline{v}_e^{\parallel}$

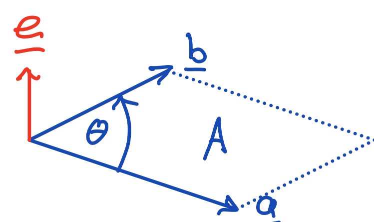


Vector product: $\underline{a}, \underline{b} \in \mathcal{V}$

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \, \hat{\underline{e}}} \qquad \theta \in [0, \pi]$$

$\hat{\underline{e}}$ unit vector $\perp$ to $\underline{a}$ & $\underline{b}$

direction right-hand rule



$|\underline{a} \times \underline{b}|$ = Area of paralelogram spanned by $\underline{a}$ & $\underline{b}$

Note: $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$ not commutative

Q: What does it mean when $\underline{a} \times \underline{b} = \underline{0}$?

$(\underline{a} \neq \underline{0}, \ \underline{b} \neq \underline{0})$

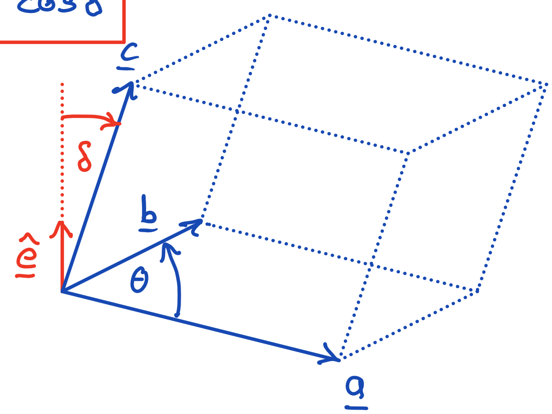
Triple scalar product $\quad \underline{a}, \underline{b}, \underline{c} \in V^3$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = |\underline{a}||\underline{b}||\underline{c}| \sin\theta \cos\delta}$$

θ angle from $\underline{a}$ to $\underline{b}$

$\hat{\underline{e}}$ right handed normal to $\underline{a}$ and $\underline{b}$

θ angle from $\hat{\underline{e}}$ to $\underline{c}$



$(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{0} \Rightarrow \underline{a}, \underline{b}, \underline{c}$ linearly dependent

$(\underline{a} \times \underline{b}) \cdot \underline{c} > 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$ form right handed system

$(\underline{a} \times \underline{b}) \cdot \underline{c} < 0 \Rightarrow \underline{a}, \underline{b}, \underline{c}$ form left handed system

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} = (\underline{b} \times \underline{c}) \cdot \underline{a} = (\underline{c} \times \underline{a}) \cdot \underline{b}}$$

$\Rightarrow$ Volume of parallelepiped spanned by $\underline{a}, \underline{b}, \underline{c}$

Q: $(\underline{a} \times b) \cdot c \stackrel{?}{=} (\underline{b} \times \underline{a}) \cdot c$

Triple vector product

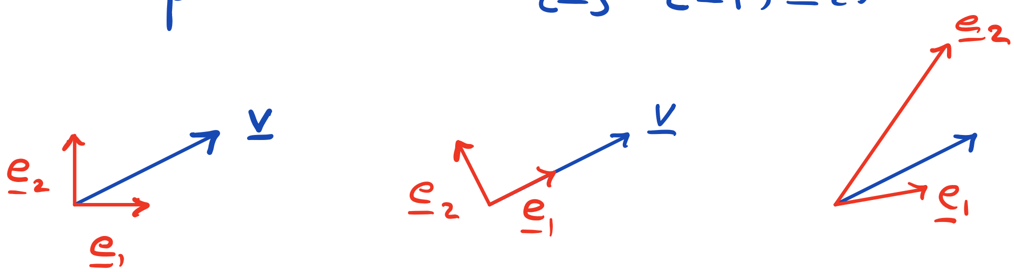
This may be new – we'll talk more about it later

$$\boxed{\begin{aligned} (\underline{a} \times \underline{b}) \times \underline{c} &= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a} \\ \underline{a} \times (\underline{b} \times \underline{c}) &= (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \end{aligned}}$$

Basis for a vector space

Def.: Basis for $\mathcal{V}$ is a set of linearly independent vectors $\{\underline{e}\}$ that span the space.

Examples in 2D: $\{\underline{e}\} = \{\underline{e}_1, \underline{e}_2)$



many choices $\Rightarrow$ not unique

In this class we will always choose a right-handed orthonormal basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

ortho-normal: $\underline{e}_1 \times \underline{e}_2 = \underline{e}_3$, $\underline{e}_2 \times \underline{e}_3 = \underline{e}_1$, $\underline{e}_3 \times \underline{e}_1 = \underline{e}_2$

right handed: $(\underline{e}_1 \times \underline{e}_2) \cdot \underline{e}_3 = 1$

$\Rightarrow$ called Cartesian reference frames

Components of a vector in a basis

Project $\underline{v}$ onto basis vectors to get components.

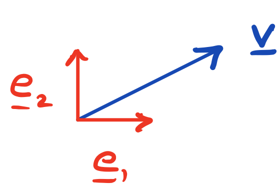
$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$$

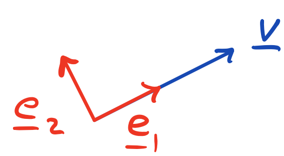
$$v_1 = \underline{v} \cdot \underline{e}_1 \qquad\qquad [\underline{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$
$$v_2 = \underline{v} \cdot \underline{e}_2$$
$$v_3 = \underline{v} \cdot \underline{e}_3$$

Here $[\underline{v}]$ is the representation of $\underline{v}$ in $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

The distinction between a vector and its representation is important for this class.

Example:





$$[\underline{v}] = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad\qquad [\underline{v}] = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

$$|\underline{v}| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad\qquad |\underline{v}| = \sqrt{\sqrt{5}^2 + 0^2} = \sqrt{5}$$

The vector is the same but representation is not.