Reynolds Transport Theorem

Let $\varphi(\underline{X},t)$ be a motion with spatial velocity field $\underline{v}(\underline{x},t)$ and Ω_t an arbitrary volume in B_t with surface $\partial\Omega_t$ and outward unit normal, $\underline{n}$. Then for any spatial scalar field $\phi(\underline{x},t)$ we have

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi \, dV_x = \int_{\Omega_t} \frac{\partial \phi}{\partial t} dV_x + \oint_{\partial\Omega_t} \phi \, \underline{v} \cdot n \, dA_x}$$

Key: Although $\Omega_t = \varphi(\Omega_0, t)$ this time derivative can be computed without knowledge of $\varphi(\underline{X},t)$.

Notice: Difficulty is that Ω_t changes with time

$\Rightarrow$ Move to reference configuration:

$$\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t) \, dV_x = \frac{d}{dt}\int_{\Omega_0} \underbrace{\phi(\varphi(\underline{X},t),t)}_{\phi_m(\underline{X},t)} J(\underline{X},t) \, dV_X$$

Ω_0 is fixed $\Rightarrow$ exchange deriv. & integral

$$= \int_{\Omega_0} \frac{d}{dt}(\phi_m J)\, dV_X = \int_{\Omega_0} \dot{\phi}_m J + \phi_m \dot{J}\, dV_X$$

where $\dot{J} = J(\nabla_x \cdot \underline{v})_m$ → show later

$$= \int_{\Omega_0} \dot{\phi}_m J + \phi_m J (\nabla_x \cdot \underline{v})_m\, dV_X$$

$$= \int_{\Omega_0} \left(\dot{\phi}_m + \phi_m (\nabla_x \cdot \underline{v})_m\right) J\, dV_X$$

$$= \int_{\Omega_t} \dot{\phi} + \phi\, \nabla_x \cdot \underline{v}\, dV_x$$

substituting spatial descripion of material deriv.

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \underbrace{\underline{v} \cdot \nabla \phi + \phi\, \nabla_x \cdot \underline{v}}_{\nabla \cdot (\phi \underline{v})}\, dV_x$$

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \underline{v})\, dV_x$$

using divergence theorem on 2^{nd} term

$$= \int_{\Omega_t} \frac{\partial \phi}{\partial t}\, dV_x + \oint_{\partial \Omega_t} \phi\, \underline{v} \cdot \underline{n}\, dA_x \checkmark$$

Where does $\dot{J}$ come from?

$$\dot{J} = J(\nabla\cdot\underline{v})_m$$

From lecture 5:

Deriv. of scalar-valued tensor fun.: $\dot{\psi}(\underline{\underline{S}}(t)) = D\psi(\underline{\underline{S}}) : \dot{\underline{\underline{S}}}$

Derivative of determinant: $D\det(\underline{\underline{S}}) = \det(\underline{\underline{S}})\,\underline{\underline{S}}^{-T}$

From lecture 3: $\underline{\underline{S}} : \underline{\underline{D}} = tr(\underline{\underline{S}}^T\underline{\underline{D}})$

$$\Rightarrow \dot{J} = \frac{d}{dt}\det(\underline{\underline{F}}) = \det(\underline{\underline{F}})\,\underline{\underline{F}}^{-T} : \dot{\underline{\underline{F}}} = \det(\underline{\underline{F}})\,tr(\underline{\underline{F}}^{-1}\dot{\underline{\underline{F}}})$$

$$\boxed{\dot{J} = \det(\underline{\underline{F}})\,tr(\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1})}$$

Material time deriv. of $F_{iJ} = \frac{\partial\varphi_i}{\partial X_J} = \varphi_{i,J}$

$$\dot{F}_{iJ} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial X_J}\varphi_i\right) = \frac{\partial}{\partial X_J}\frac{\partial}{\partial t}\varphi_i = \frac{\partial}{\partial X_J}V_i(\underline{X},t) = \frac{\partial}{\partial X_J}v_i(\underline{\varphi}(\underline{X},t),t)$$

where $\frac{\partial}{\partial X_J} = \frac{\partial}{\partial x_k}\frac{\partial x_k}{\partial X_J} = \frac{\partial}{\partial x_k}\varphi_{k,J} = \frac{\partial}{\partial x_k}F_{k,J}$

$$\Rightarrow \dot{F}_{iJ} = \frac{\partial}{\partial x_k}v_i(\underline{x},t)\Big|_{\underline{x}=\underline{\varphi}(\underline{X},t)}F_{kJ}$$

$$= v_{i,k}(\underline{x},t)\Big|_{\underline{x}=\varphi(\underline{X},t)}F_{kJ}$$

$$\boxed{\dot{\underline{\underline{F}}} = \nabla_x\underline{v}\Big|_{\underline{x}=\varphi(\underline{X},t)}\underline{\underline{F}}}$$

$$\dot{J} = J \,\mathrm{tr}(\dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}) = J\,\mathrm{tr}(\dot{\underline{\underline{F}}}) = J\,\mathrm{tr}(\nabla_x \underline{v})\big|_{\underline{x}=\varphi(\underline{X},t)}$$

$$= J\,\nabla_x \cdot \underline{v}\big|_{\underline{x}=\varphi(\underline{X},t)}$$

$$\Rightarrow \quad \dot{J} = J\,\nabla_x \cdot \underline{v}\big|_{\underline{x}=\varphi(\underline{X},t)}$$