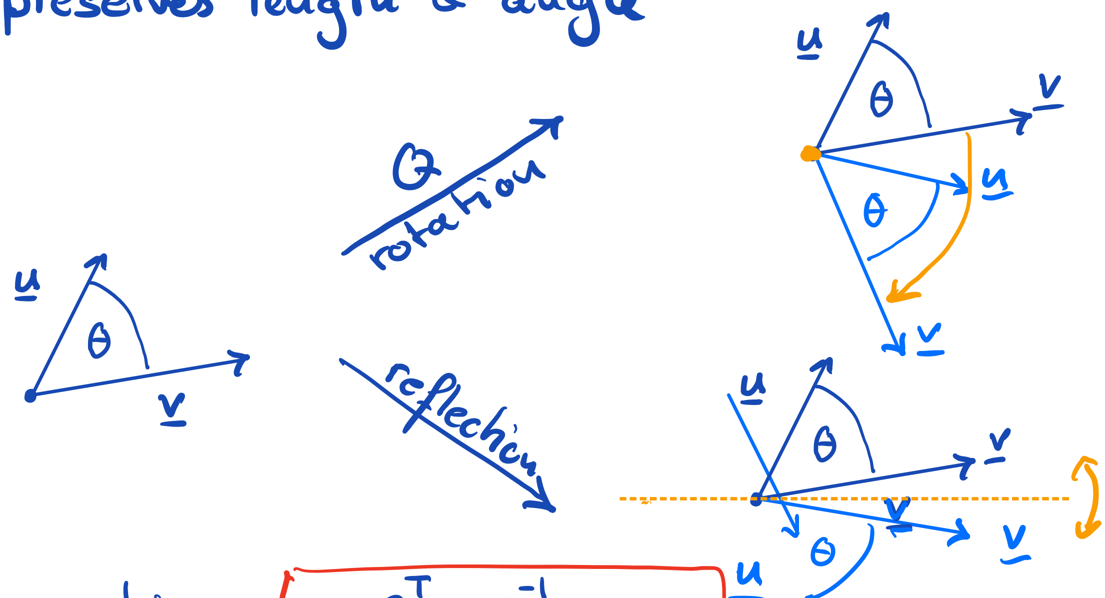
Orthogonal tensors

An orthogonal tensor $\mathbf{Q} \in \mathcal{V}^2$ is a linear transformation satisfying

$$\boxed{\mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}} \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{V}$$

$\Rightarrow$ preserves length & angle



Properties:

$$\boxed{\begin{gathered} \mathbf{Q}^T = \mathbf{Q}^{-1} \\ \mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I} \\ \det(\mathbf{Q}) = \pm 1 \end{gathered}}$$

$$\det(\mathbf{I}) = 1 \Rightarrow \det(\mathbf{Q}^T\mathbf{Q}) = \det(\mathbf{Q}^T)\det(\mathbf{Q}) = \det(\mathbf{Q})^2 = 1$$

If $\det(\mathbf{Q}) = 1 \Rightarrow$ rotation

$\det(\mathbf{Q}) = -1 \Rightarrow$ reflection

Change in basis

Both $\underline{v} \in \mathcal{V}$ and $\underline{\underline{S}} \in \mathcal{V}^2$ are invariant upon change of basis, but their representations $[\underline{v}]$ and $[\underline{\underline{S}}]$ change.

Consider two frames $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$

Representation of $\underline{e}'_j$ in $\{\underline{e}_i\}$ is

$$\underline{e}'_j = (\underline{e}'_j \cdot \underline{e}_1)\,\underline{e}_1 + (\underline{e}'_j \cdot \underline{e}_2)\,\underline{e}_2 + (\underline{e}'_j \cdot \underline{e}_3)\,\underline{e}_3$$

$$= (\underline{e}'_j \cdot \underline{e}_i)\,\underline{e}_i$$

$$\underline{e}'_j = A_{ij}\,\underline{e}_i \qquad \text{where} \quad A_{ij} = \underline{e}_i \cdot \underline{e}'_j$$

note transpose

Here $\underline{\underline{A}}$ is the change of basis tensor

$$\boxed{\underline{\underline{A}} = A_{ij}\,\underline{e}_i \otimes \underline{e}_j \qquad A_{ij} = \underline{e}_i \cdot \underline{e}'_j}$$

Similarly we can express $\underline{e}_i$ in $\{\underline{e}'_k\}$

$$\underline{e}_i = (\underline{e}_i \cdot \underline{e}'_k)\,\underline{e}'_k = A_{ik}\,\underline{e}'_k$$

We have $\underline{e}'_j = A_{ij}\,\underline{e}_i \qquad \underline{e}_i = A_{ik}\,\underline{e}'_k$

$$\underline{e}'_j = \underbrace{A_{ij}\,A_{ik}}\,\underline{e}'_k$$
$$\underline{e}'_j = \delta_{jk}\,\underline{e}'_k$$

$$A_{ij}\,A_{ik} = \delta_{jk}$$

$$\underline{e}_i = A_{ik}\left(A_{lk}\,\underline{e}_l\right)$$
$$= A_{ik}\,A_{lk}\,\underline{e}_l$$
$$\underline{e}_i = \delta_{il}\,\underline{e}_l$$
$$A_{ik}\,A_{lk} = \delta_{il}$$

$\Rightarrow \quad \boxed{\underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{I}}} \qquad \underline{\underline{A}}$ is orthogonal

If both $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ are right-handed

then $\underline{\underline{A}}$ must be a rotation, $\det(\underline{\underline{A}}) = 1$

Change in representation

Consider $\underline{v} \in \mathcal{V}$ and $\underline{\underline{S}} \in \mathcal{V}^2$

with representations:

$[\underline{v}]$ and $[\underline{\underline{S}}]$ in $\{\underline{e}_i\}$

$[\underline{v}]'$ and $[\underline{\underline{S}}]'$ in $\{\underline{e}'_i\}$

so that $[\underline{v}] \neq [\underline{v}]'$ and $[\underline{\underline{S}}] \neq [\underline{\underline{S}}]'$

then $\boxed{[\underline{v}] = [\underline{\underline{A}}][\underline{v}]' \quad \text{and} \quad [\underline{v}]' = [\underline{\underline{A}}]^T [\underline{v}]}$

to see this

$$\underline{v} = v_i \underline{e}_i = v_j' \underline{e}_j' \qquad \text{where} \quad \underline{e}_j' = A_{ij} \underline{e}_i$$

$$v_i \underline{e}_i = v_j' A_{ij} \underline{e}_i \quad \Rightarrow \quad v_i = A_{ij} v_j' \ \checkmark$$

Similarly $\boxed{\begin{array}{l} [\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T \\ [\underline{\underline{S}}]' = [\underline{\underline{A}}]^T[\underline{\underline{S}}][\underline{\underline{A}}] \end{array}}$ $\Rightarrow$ HW2

In summary; Change of basis is a rotation and can hence be written as an orthogonal tensor $\underline{\underline{A}}$ with components $A_{ij} = \underline{e}_i \cdot \underline{e}_j'$.

Next we show that $tr(\underline{\underline{S}})$ and $det(\underline{\underline{S}})$ are invariant under change in basis

$\Rightarrow$ important for constitutive laws

Invariance of trace

For $\underline{\underline{S}} \in \mathcal{V}^2$ with $[\underline{\underline{S}}]$ in $\{e_i\}$ and $[\underline{\underline{S}}]'$ in $\{e_i'\}$

$$\boxed{\mathrm{tr}[\underline{\underline{S}}] = \mathrm{tr}[\underline{\underline{S}}]'}$$

consider

$[\underline{\underline{S}}] = [\underline{\underline{A}}][\underline{\underline{S}}]'[\underline{\underline{A}}]^T$ or $[\underline{\underline{S}}]_{ij} = [\underline{\underline{A}}]_{ik}[\underline{\underline{S}}]'_{kl}[A]_{jl}$

$$\begin{aligned}
\mathrm{tr}[\underline{\underline{S}}] = [\underline{\underline{S}}]_{ii} &= [\underline{\underline{A}}]_{ik}\,[\underline{\underline{S}}]'_{kl}\,[\underline{\underline{A}}]_{il} \\
&= [\underline{\underline{S}}]'_{kl}\,[\underline{\underline{A}}]_{ik}\,[\underline{\underline{A}}]_{il} \\
&= [\underline{\underline{S}}]'_{kl}\,\underbrace{\delta_{kl}} = [\underline{\underline{S}}]'_{kk} = \mathrm{tr}[\underline{\underline{S}}]'
\end{aligned}$$

Invariance of determinant

For $\underline{\underline{S}} \in \mathcal{V}^2$ with $[\underline{\underline{S}}]$ in $\{e_i\}$ and $[\underline{\underline{S}}]'$ in $\{e_i'\}$

$$\boxed{\det[\underline{\underline{S}}] = \det[\underline{\underline{S}}]'} \Rightarrow \text{HW 2}$$

Eigenvalues & vectors of tensors

By the eigen pair of $\underline{\underline{S}} \in \mathcal{V}^2$ we mean the scalar λ and the vector $\underline{v}$ such that

$$\underline{\underline{S}}\,\underline{v} = \lambda\,\underline{v}$$

λ = eigenvalue $\qquad$ $\underline{v}$ = eigenvector

λ's are roots of characteristic polynomial

$$p(\lambda) = \det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = 0$$

For each λ_p we one ore more $\underline{v}_p$ satisfying

$$(\underline{\underline{S}} - \lambda_p \underline{\underline{I}})\,\underline{v}_p = \underline{0}$$

In continuum mechanics we are mostly concerned with symmetric tensors, eg. stress.

Eigen problem for symmetric tensors

1) All λ_p's real
2) All λ_p's are positive ($\underline{\underline{S}}$ sym. pos. def.)
3) All $\underline{v}_p$'s corresponding to distinct λ_p's are orthogonal

$\underline{\underline{S}}$ is symmetric positive definite (spd)

if $\quad \underline{v} \cdot \underline{\underline{S}}\, \underline{v} > 0 \quad$ for all $\underline{v} \in \mathcal{V}$

by def. of eigenpair $\underline{\underline{S}}\, \underline{v} = \lambda \underline{v}$

$$\underline{v} \cdot \lambda \underline{v} > 0$$

$$\lambda |v|^2 > 0 \quad \rightarrow \quad \lambda > 0$$

For orthogonality consider two eigenpairs $(\lambda, \underline{v})$ and $(\omega, \underline{u})$ so that $\lambda \neq \omega$

$$\underline{\underline{S}}\,\underline{v} = \lambda \underline{v} \quad \text{and} \quad \underline{\underline{S}}\,\underline{u} = \omega \underline{u}$$

Consider $\lambda(\underline{v}\cdot\underline{u}) = \underline{\underline{S}}\,\underline{v} \cdot \underline{u} = \underline{v} \cdot \underline{\underline{S}}^T \underline{u}$

$$= \underline{v} \cdot \underline{\underline{S}}\,\underline{d} = \omega(\underline{v}\cdot\underline{u})$$

$$\lambda(\underline{v}\cdot\underline{u}) = \omega(\underline{v}\cdot\underline{u})$$

since $\lambda \neq \omega \Rightarrow \underline{v}\cdot\underline{u} = \underline{0}$

Spectral decomposition

If $\underline{\underline{S}} = \underline{\underline{S}}^T$ there exists a frame $\{\underline{e}_i\}$ consisting of the eigenvectors of $\underline{\underline{S}}$ so that

$$\boxed{\underline{\underline{S}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i}$$

To see this consider the following

Since $\underline{v}_i$ are orthonormal : $\underline{\underline{I}} = \underline{v}_i \otimes \underline{v}_i$

$$\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}(\underline{v}_i \otimes \underline{v}_i) = (\underline{\underline{A}}\,\underline{v}_i) \otimes v_i = \sum_{i=1}^{3} (\lambda_i v_i) \otimes v_i$$

$$= \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i$$

where we have used : $\underline{\underline{A}}(\underline{u} \otimes \underline{v}) = (\underline{\underline{A}}\,\underline{u}) \otimes \underline{v}$ $\Rightarrow$ HW2

$$\alpha(\underline{u} \otimes \underline{v}) = (\alpha \underline{u}) \otimes \underline{v}$$

Representation in eigenframe

$$[\underline{\underline{S}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ diagonalize tensor

The principal invariants of $\underline{\underline{S}} \in \mathcal{V}^2$ are

$$\boxed{\begin{aligned} I_1(\underline{\underline{S}}) &= tr(\underline{\underline{S}}) = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2(\underline{\underline{S}}) &= \tfrac{1}{2}\left((tr\,\underline{\underline{S}})^2 - tr(\underline{\underline{S}}^2)\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ I_3(\underline{\underline{S}}) &= det(\underline{\underline{S}}) = \lambda_1\lambda_2\lambda_3 \end{aligned}}$$

These 3 scalars are frame invariant.

Set of invariants $I_S = \{I_i(\underline{\underline{S}})\}$

Rewrite the char. polynomial with invariants

$$\boxed{\det(\underline{\underline{S}} - \lambda \underline{\underline{I}}) = -\lambda^3 + I_1(\underline{\underline{S}})\lambda^2 - I_2(\underline{\underline{S}})\lambda + I_3(\underline{\underline{S}}) = 0}$$

This is most easily shown in eigen frame
just collect terms in powers of λ.

Consider $\underline{\underline{S}}\,\underline{\underline{S}}\,\underline{v} = \underline{\underline{S}}(\lambda \underline{v}) = \lambda^2 \underline{v}$

in general $\underline{\underline{S}}^a \underline{v} = \lambda^a \underline{v}$

multiplying char. poly. by $\underline{v}$

$$-\lambda^3 \underline{v} + I_1(\underline{\underline{S}})\lambda^2 \underline{v} - I_2(\underline{\underline{S}})\lambda \underline{v} + I_3(\underline{\underline{S}})\underline{v} = 0$$

$$-\underline{\underline{S}}^3 \underline{v} + I_1(\underline{\underline{S}})\underline{\underline{S}}^2 \underline{v} - I_2(\underline{\underline{S}})\underline{\underline{S}}\,\underline{v} + I_3(\underline{\underline{S}})\underline{v} = 0$$

since $\underline{v} \neq 0$

$$\boxed{-\underline{\underline{S}}^3 + I_1(\underline{\underline{S}})\underline{\underline{S}}^2 - I_2(\underline{\underline{S}})\underline{\underline{S}} + I_3(\underline{\underline{S}}) = 0}$$

every tensor satisfies its char. poly.

Cayley-Hamilton Theorem

Tensor square root

If $\underline{\underline{C}}$ is a s.p.d. tensor with eienpair $(\lambda, \underline{v})$

then there is a unique tensor $\underline{\underline{U}} = \sqrt{C}$

$$\boxed{\underline{\underline{U}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, \underline{e}_i \otimes \underline{e}_i}$$

Polar decomposition

Any tensor $\underline{\underline{F}} \in \mathcal{V}^2$ with $\det(\underline{\underline{F}}) > 0$

has a right and left polar decomposition

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$$

where $\underline{\underline{U}} = \sqrt{F^T F}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d

and $\underline{\underline{R}}$ is a rotation.

To see this consider

$\det(\underline{\underline{F}}) > 0 \Rightarrow \underline{\underline{F}}\,\underline{v} \neq \underline{0}$ for $\underline{v} \neq \underline{0}$

$\det(\underline{\underline{F}}^T) > 0 \Rightarrow \underline{\underline{F}}^T \underline{v} \neq \underline{0}$ for $\underline{v} \neq \underline{0}$

To show $\underline{\underline{U}}$ & $\underline{\underline{V}}$ are spd

Clearly: $(\underline{\underline{F}}\,\underline{v}) \cdot (\underline{\underline{F}}\,\underline{v}) > 0$

$$(\underline{\underline{F}}\,\underline{v})^T(\underline{\underline{F}}\,\underline{v}) = \underline{v}^T\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{v} = \underline{v}\cdot\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{v} > 0$$

Similarly: $(\underline{\underline{F}}^T\underline{v}) \cdot (\underline{\underline{F}}^T\underline{v}) > 0$

$$(\underline{\underline{F}}^T\underline{v})^T(\underline{\underline{F}}^T\underline{v}) = \underline{v}^T\underline{\underline{F}}\,\underline{\underline{F}}^T\underline{v} = \underline{v}\cdot\underline{\underline{F}}\,\underline{\underline{F}}^T\underline{v} > 0$$

$\Rightarrow$ $\underline{\underline{F}}^T\underline{\underline{F}}$ and $\underline{\underline{F}}\,\underline{\underline{F}}^T$ are s.p.d.

so that we can define $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$

$$\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$$

Show that $\underline{\underline{R}}$ is rotation

Show $\det(\underline{\underline{R}}) > 0$

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \qquad \underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1} \qquad \Rightarrow \det(\underline{\underline{R}}) = \frac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$$

Show $\underline{\underline{R}}$ is orthonormal

$$\underline{\underline{R}}^T\underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underbrace{\underline{\underline{U}}^{-T}}_{\underline{\underline{U}}^{-1}}\underbrace{\underline{\underline{F}}^T\underline{\underline{F}}}_{\underline{\underline{U}}^{2}}\,\underline{\underline{U}}^{-1} = \underline{\underline{U}}^{-1}\underline{\underline{U}}^{2}\underline{\underline{U}}^{-1} = \underline{\underline{I}}$$

Similar arguments hold for $\underline{\underline{F}} = \underline{\underline{V}}\,\underline{\underline{R}}$