Differentiation of Tensor fields

A field is a function of space.

scalar fields: $\phi(\underline{x})$ temp., density

vector fields: $\underline{v}(\underline{x})$ force, velocity

tensor fields: $\underline{\underline{S}}(\underline{x})$ stress, conductivity

Today's lecture is review and extension of concepts from multivariable calculus.

Gradients

Gradient of scalar field

Scalar field $\phi(\underline{x})$ is differentiable at $\underline{x}$ if there exists a vector field $\nabla\phi \in \mathcal{V}$ such that

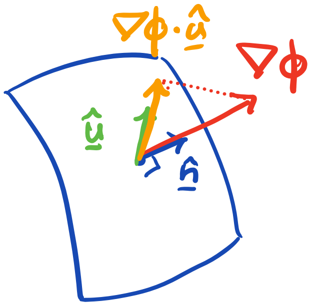
$$\boxed{\phi(\underline{x}+\underline{h}) = \phi(\underline{x}) + \nabla\phi(\underline{x}) \cdot \underline{h} + \mathcal{O}(|\underline{h}|)}$$

by Taylor expansion. Or equivalently

$$\boxed{\nabla\phi(\underline{x}) \cdot \underline{\hat{u}} = \frac{d}{d\varepsilon}\, \phi(\underline{x} + \epsilon\underline{\hat{u}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{v} \in \mathcal{V}$$

where $\underline{h} = \epsilon\,\underline{\hat{u}}$ and $|\underline{\hat{u}}| = 1$.

The vector $\nabla\phi$ is called the gradient of ϕ.



Consider a level set of ϕ

$\nabla\phi \parallel \hat{\underline{n}}$ in direction of increasing ϕ

$$\hat{\underline{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

Directional derivative (Gâteaux operator)

$$\boxed{D_{\hat{\underline{u}}}\phi(\underline{x}) = \frac{d}{d\epsilon}\,\phi(\underline{x}+\epsilon\,\hat{\underline{u}})\Big|_{\epsilon=0} = \nabla\phi(\underline{x})\cdot\hat{\underline{u}}}$$

Representation of the gradient in frame $\{\underline{e}_i\}$

$$\phi(\bar{\underline{x}}+\epsilon\,\underline{u}) = \phi(\underbrace{\bar{x}_1+\epsilon u_1}_{x_1},\ \underbrace{\bar{x}_2+\epsilon u_2}_{x_2},\ \underbrace{\bar{x}_3+\epsilon u_3}_{x_3})$$

$$\nabla\phi\cdot\hat{\underline{u}} = \frac{d}{d\epsilon}\,\phi(\bar{x}_1+\epsilon u_1,\ \bar{x}_2+\epsilon u_2,\ \bar{x}_3+\epsilon u_3)\Big|_{\epsilon=0}$$

$$= \frac{d\phi}{dx_1}\frac{dx_1}{d\epsilon} + \frac{d\phi}{dx_2}\frac{dx_2}{d\epsilon} + \frac{d\phi}{dx_3}\frac{dx_3}{d\epsilon}\Big|_{\epsilon=0}$$

$$= \frac{d\phi}{dx_1}u_1 + \frac{d\phi}{dx_2}u_2 + \frac{d\phi}{dx_3}u_3$$

$$= \frac{\partial\phi}{\partial x_i}u_i = \phi_{,i}\,u_i = \phi_{,i}\,u_j\,\delta_{ij} = \phi_{,i}\,u_j\,\underline{e}_i\cdot\underline{e}_j$$

$$\nabla\phi\cdot\hat{\underline{u}} = (\phi_{,i}\,\underline{e}_i)\cdot(u_j\,\underline{e}_j) \quad ✓$$

Note: Index notation for derivatives

$$\boxed{\frac{\partial \phi}{\partial x_i} = \phi_{,i}}$$ derivative index after comma!

Gradient in components: $[\nabla \phi] = \phi_{,i}\,\underline{e}_i = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{pmatrix}$

Gradient of a vector field

A vector field $\underline{v}(x) \in \mathcal{V}$ is differentiable at $\underline{x}$ if there exists a tensor field $\nabla \underline{v}(\underline{x}) \in \mathcal{V}^2$ such that

$$\underline{v}(\underline{x}+\underline{h}) = \underline{v}(\underline{x}) + \nabla \underline{v}(x)\,\underline{h} + o(|h|)$$

by Taylor expansion or equivalently

$$\boxed{\nabla \underline{v}\,\hat{\underline{u}} = \frac{d}{d\epsilon}\underline{v}(\underline{x}+\epsilon\hat{\underline{u}})\Big|_{\epsilon=0}} \quad \text{for all } \underline{u} \in \mathcal{V}$$

where $\underline{h} = \epsilon \hat{\underline{u}}$

In frame $\{\underline{e}_i\}$ we write components of $\underline{v}$ as $v_i = v_i(x_1, x_2, x_3)$. For any scalar ϵ and unit vector $\hat{\underline{u}} = u_k \underline{e}_k$ at $\bar{\underline{x}} = \bar{x}_j \underline{e}_j$

we have the i-th component

$$v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}}) = v_i(\bar{x}_1 + \epsilon u_1, \bar{x}_2 + \epsilon u_2, \bar{x} + \epsilon u_3)$$

by the chain rule

$$\frac{d}{d\epsilon} v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}}) = \frac{\partial v_i}{\partial x_1} u_1 + \frac{\partial v_i}{\partial x_2} u_2 + \frac{\partial v_i}{\partial x_3} u_3 = \frac{\partial v_i}{\partial x_j} u_j$$

For full vector $\underline{v} = v_i \underline{e}_i$

$$\nabla \underline{v}\, \underline{\hat{u}} = \frac{d}{d\varepsilon} \underline{v}(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\Big|_{\epsilon=0} = \frac{d}{d\varepsilon}\left(v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\underline{e}_i\right)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\varepsilon}\left(v_i(\underline{\bar{x}} + \epsilon \underline{\hat{u}})\right)\Big|_{\epsilon=0} \underline{e}_i = \frac{\partial v_i}{\partial x_j} u_j \underline{e}_i$$

components: $\boxed{[\nabla \underline{v}]_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}}$

Representation $\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$

Explicit

$$\nabla \underline{v} = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix} = \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \\ \nabla v_3^T \end{bmatrix}$$

Divergence of a vector field

Def: To any $\underline{v}(x) \in \mathcal{V}$ we associate a scalar field $\nabla \cdot \underline{v}$ called the divergence of $\underline{v}$

$$\boxed{\nabla \cdot \underline{v} = \mathrm{tr}(\nabla \underline{v})}$$

In frame $\{\underline{e}_i\}$ with $\underline{v}(\underline{x}) = v_i(\underline{x})\, \underline{e}_i$ we have

$$\boxed{\nabla \cdot \underline{v} = \mathrm{tr}(\nabla \underline{v}) = v_{i,i}}$$

If $\nabla \cdot \underline{v} = 0$ a field is solenoidal or divergence free. If $\underline{v}$ is a displacement or velocity then $\nabla \cdot \underline{v}$ is related to (rate of) volume change.

Divergence of a tensor field

To any $\underline{\underline{S}}(\underline{x}) \in \mathcal{V}^2$ we associate a vector field $\nabla \cdot \underline{\underline{S}} \in \mathcal{V}$ called the divergence of $\underline{\underline{S}}$

$$\boxed{(\nabla \cdot \underline{\underline{S}}) \cdot \underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a})} \quad \text{for all } \underline{a} \in \mathcal{V}$$

uses definition of vector divergence!

In frame $\{\underline{e}_i\}$ with $\underline{\underline{S}} = S_{ij}\, \underline{e}_i \otimes \underline{e}_j$ and $\underline{a} = a_k \underline{e}_k$

we have $\underline{q} = \underline{\underline{S}}^T \underline{a}$ or $q_j = S_{ij} a_i$ $(q_i = S_{ji} q_j)$

substituting

$$(\nabla \cdot \underline{\underline{S}})\underline{a} = \nabla \cdot (\underline{\underline{S}}^T \underline{a}) = \nabla \cdot \underline{q} = \mathrm{tr}(\nabla \underline{q}) = q_{j,j}$$

$$= S_{ij,j}\, a_i = (S_{ij,j}\, \underline{e}_i) \cdot (a_k \underline{e}_k)$$

by the arbitraryness of $\underline{a}$ we have

$$\boxed{\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i}$$

Gradient & Divergence product rules

$\phi \in \mathbb{R}$, $\underline{v} \in \mathcal{V}$, $\underline{\underline{S}} \in \mathcal{V}^2$

$$\boxed{\begin{aligned}
\nabla \cdot (\phi \underline{v}) &= \underline{v} \cdot \nabla \phi + \phi \nabla \cdot \underline{v} \\
\nabla \cdot (\phi \underline{\underline{S}}) &= \underline{\underline{S}} \nabla \phi + \phi \nabla \cdot \underline{\underline{S}} \\
\nabla \cdot (\underline{\underline{S}}^T \underline{v}) &= (\nabla \cdot \underline{\underline{S}}) \cdot \underline{v} + \underline{\underline{S}} : \nabla \underline{v} \\
\nabla (\phi \underline{v}) &= \underline{v} \otimes \nabla \phi + \phi \nabla \underline{v}
\end{aligned}}$$

Note: Last identity is gradient.

Example: $\nabla\cdot(\underline{\underline{S}}^T\underline{v})$ note $\underline{\underline{S}}=\underline{\underline{S}}(x)$ and $\underline{v}=v(\underline{x})$

$$q(x)=\underline{\underline{S}}^T(\underline{x})\,\underline{v}(\underline{x}) \qquad q_j = S_{ij}v_i$$

$$\nabla\cdot q = tr(q) = q_{j,j} = (S_{ij}v_i)_{,j}$$

$$= S_{ij,j}v_i + S_{ij}v_{i,j}$$

$$=(\nabla\cdot\underline{\underline{S}})\cdot\underline{v} + \underline{\underline{S}}:\nabla\underline{v} \quad \checkmark$$

Example: $\nabla(\phi\underline{v}) = (\phi v_i)_{,j}\,\underline{e}_i\otimes\underline{e}_j$

$$=(\phi_{,j}v_i + \phi\, v_{i,j})\,\underline{e}_i\otimes\underline{e}_j$$

$$= v_i\phi_{,j}\,\underline{e}_i\otimes\underline{e}_j + \phi\, v_{i,j}\,\underline{e}_i\otimes\underline{e}_j$$

$$= \underline{v}\otimes\nabla\phi + \phi\nabla\underline{v} \quad \checkmark$$

Curl of a vector field

To any $\underline{v}(\underline{x})\in\mathcal{V}$ we associate another vector field $\nabla\times\underline{v}$ defined by

$$\boxed{(\nabla\times\underline{v})\times\underline{a} = (\nabla\underline{v} - \nabla\underline{v}^T)\,\underline{a}} \quad \text{for all } \underline{a}\in\mathcal{V}$$

Here $\underline{w} = \nabla \times \underline{v}$ is the axial vector of

$$\underline{\underline{T}} = \nabla \underline{v} - \nabla \underline{v}^T = 2\,\text{skew}(\nabla \underline{v})$$

In index notation

$$w_j = \tfrac{1}{2}\epsilon_{ijk} T_{ik} = \tfrac{1}{2}\epsilon_{ijk}(v_{i,k} - v_{k,i})$$

$$= \tfrac{1}{2}(\epsilon_{ijk} v_{i,k} - \epsilon_{ijk} v_{k,i}) \qquad \epsilon_{ijk} = -\epsilon_{kji}$$

$$= \tfrac{1}{2}(\epsilon_{ijk} v_{i,k} + \epsilon_{kji} v_{k,i}) \qquad \text{flip } i \leftrightarrow k \text{ in second}$$

$$w_j = \epsilon_{ijk} v_{i,k}$$

$$\Rightarrow \quad \boxed{\underline{w} = \nabla \times \underline{v} = \epsilon_{ijk} v_{i,k} \underline{e}_j}$$

Note: Equivalently $\nabla \times \underline{v} = -\epsilon_{ijk} v_{i,j} \underline{e}_k$

by switching & renaming indices

Explicitly: $\nabla \times \underline{v} = (v_{3,2} - v_{2,3})\underline{e}_1 + (v_{1,3} - v_{3,1})\underline{e}_2$

$$+ (v_{2,1} - v_{1,2})\underline{e}_3$$

Physical interpretation:

If $\underline{v}$ is a velocity field then $\nabla \times \underline{v}$ measures the angular velocity.

If $\nabla \times \underline{v} = 0 \Rightarrow \underline{v}(x)$ is irrotational/conservative

Further we can show

$\boxed{\nabla \times \nabla \phi = \underline{0}}$ and $\boxed{\nabla \cdot (\nabla \times \underline{v}) = 0} \Rightarrow$ HW3

this follows as

$$\nabla \times \nabla \phi = \nabla \times (\phi_{,i}\, \underline{e}_i) = \epsilon_{ijk} (\phi_{,i})_{,k}\, \underline{e}_j$$

$$= \epsilon_{ijk}\, \phi_{,ik}\, \underline{e}_j$$

$$= \tfrac{1}{2} \left(\epsilon_{ijk}\, \phi_{,ik} + \epsilon_{ijk}\, \phi_{,ik} \right) \underline{e}_j$$

2nd term $\epsilon_{ijk} = -\epsilon_{kji}$

$$= \tfrac{1}{2} \left(\epsilon_{ijk}\, \phi_{,ik} - \epsilon_{kji}\, \phi_{,ik} \right) \underline{e}_j$$

$\phi_{,ik} = \phi_{,ki}$

$$= \tfrac{1}{2} \left(\epsilon_{ijk}\, \phi_{,ik} - \epsilon_{kji}\, \phi_{,ki} \right) \underline{e}_j$$

rename dummys in second term $i \Leftrightarrow j$

$$= \tfrac{1}{2} \left(\epsilon_{ijk}\, \phi_{,ik} - \epsilon_{ijk}\, \phi_{,ik} \right) \underline{e}_j$$

$$= \underline{0}$$

Laplacian

To any scalar field $\phi \in \mathbb{R}$ we associate another scalar field $\Delta\phi = \nabla^2\phi$ defined by

$$\boxed{\Delta\phi = \nabla^2\phi = \nabla\cdot\nabla\phi}$$

In frame $\{\underline{e}_i\}$ with $\nabla\phi = \phi_{,i}\,\underline{e}_i$ we have

$$\nabla\cdot\nabla\phi = \mathrm{tr}(\nabla\nabla\phi) = \mathrm{tr}(\phi_{,ij}\,\underline{e}_i\otimes\underline{e}_j) = \phi_{,ii}$$

$$\boxed{\nabla^2\phi = \phi_{,ii}}$$

Scalar Laplacian governs steady heat flow.

To any vector field $\underline{v}(x) \in \mathcal{V}$ we associate another vector field $\Delta\underline{v} = \nabla^2\underline{v} \in \mathcal{V}$ defined by $\boxed{\Delta\underline{v} = \nabla^2\underline{v} = \nabla\cdot\nabla\underline{v}}$

In frame $\{\underline{e}_i\}$ with $\underline{v} = v_i \underline{e}_i$, $\nabla \underline{v} = v_{i,j}\, \underline{e}_i \otimes \underline{e}_j$

and $\nabla \cdot \underline{\underline{S}} = S_{ij,j}\, \underline{e}_i$ we have

$$\boxed{\Delta \underline{v} = v_{i,jj}\, \underline{e}_i}$$

Vector Laplacian governs viscous flow.

There are several useful identities. One commonly used relation

$$\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v})$$

if $\underline{v}(x)$ is both solenoidal ($\nabla \cdot \underline{v} = 0$) and irrotational ($\nabla \times \underline{v} = 0$) then $\nabla^2 \underline{v} = 0$ and $\underline{v}$ is harmonic.

Used in derivation of incompressible Navier-Stokes eqn.