Polar decomposition

Any tensor $\underline{\underline{F}} \in \mathcal{V}^2$ with $\det(\underline{\underline{F}}) > 0$ has a right and left polar decomposition

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}}$$

where $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d and $\underline{\underline{R}}$ is a rotation.

To see this consider

$$\det(\underline{\underline{F}}) > 0 \quad \Rightarrow \quad \underline{\underline{F}}\,\underline{v} \neq \underline{0} \quad \text{for} \quad \underline{v} \neq \underline{0}$$

$$\det(\underline{\underline{F}}^T) > 0 \quad \Rightarrow \quad \underline{\underline{F}}^T\underline{v} \neq \underline{0} \quad \text{for} \quad \underline{v} \neq \underline{0}$$

To show $\underline{\underline{U}}$ & $\underline{\underline{V}}$ are spd

Clearly: $(\underline{\underline{F}}\,\underline{v}) \cdot (\underline{\underline{F}}\,\underline{v}) > 0$

$$(\underline{\underline{F}}\,\underline{v})^T(\underline{\underline{F}}\,\underline{v}) = \underline{v}^T\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{v} = \underline{v} \cdot \underline{\underline{F}}^T\underline{\underline{F}}\,\underline{v} > 0$$

Similarly: $(\underline{\underline{F}}^T\underline{v}) \cdot (\underline{\underline{F}}^T\underline{v}) > 0$

$$(\underline{\underline{F}}^T\underline{v})^T(\underline{\underline{F}}^T\underline{v}) = \underline{v}^T\underline{\underline{F}}\,\underline{\underline{F}}^T\underline{v} = \underline{v} \cdot \underline{\underline{F}}\,\underline{\underline{F}}^T\underline{v} > 0$$

$\Rightarrow$ $\underline{\underline{F}}^T \underline{\underline{F}}$ and $\underline{\underline{F}}\,\underline{\underline{F}}^T$ are s.p.d.

so that we can define $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$

$\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$

Show that $\underline{\underline{R}}$ is rotation

Show $\det(\underline{\underline{R}}) > 0$

$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$ $\quad$ $\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$ $\quad \Rightarrow \det(\underline{\underline{R}}) = \dfrac{\det(\underline{\underline{F}})}{\det(\underline{\underline{U}})} > 0$

Show $\underline{\underline{R}}$ is orthonormal

$$\underline{\underline{R}}^T \underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T (\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underbrace{\underline{\underline{U}}^{-T}}_{\underline{\underline{U}}^{-1}} \underbrace{\underline{\underline{F}}^T \underline{\underline{F}}}_{\underline{\underline{U}}^2} \underline{\underline{U}}^{-1} = \underline{\underline{U}}^{-1} \underline{\underline{U}}^2 \underline{\underline{U}}^{-1} = \underline{\underline{I}}$$

Similar arguments hold for $\underline{\underline{F}} = \underline{\underline{V}}\,\underline{\underline{R}}$

<u>Tensor square root</u>

If $\underline{\underline{C}}$ is a s.p.d. tensor with ei en pair $(\lambda, \underline{v})$

then there is a unique tensor $\underline{\underline{U}} = \sqrt{C}$

$$\boxed{\underline{\underline{U}} = \sum_{i=1}^{3} \sqrt{\lambda_i}\, \underline{e}_i \otimes \underline{e}_i}$$

Analysis of local deformation

Any $\varphi(\underline{X})$ can be approximated locally as a homogeneous affine deformation.

$$\boxed{\underline{x} = \underline{\varphi}(\underline{X}) = \underline{c} + \underline{\underline{F}}\,\underline{X}} \quad \text{where} \quad \underline{\underline{F}} = \nabla \underline{\varphi}$$

$\underline{\underline{F}}$ is a measure of strain but it is not suitable as strain tensor, because it contains rotations that do not lead to deformation.

Building strain tensor is 3 step process

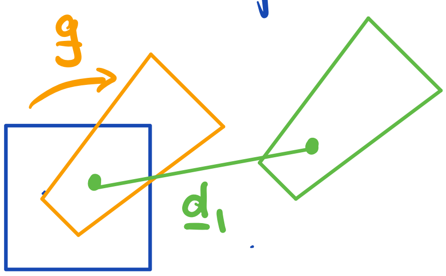
1) Remove translations
2) Remove rotations
3) Find principal streches

1) Translation - fixed point decomposition

Any hom. φ can be decomposed as

$$\boxed{\varphi = \underline{d}_1 \circ g = g \circ \underline{d}_2}$$

where $g = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ is a hom. def with fixed point $\underline{Y}$ and $\underline{d}_i = \underline{X} + \underline{a}_i$ with $i = \{1, 2\}$ are translations from $\underline{Y}$.



Consider points $\underline{X}$ and $\underline{Y}$ and their maps

$$\underline{x} = \underline{c} + \underline{\underline{F}}\,\underline{X} \qquad \text{and} \qquad \underline{y} = \underline{c} + \underline{\underline{F}}\,\underline{Y}$$

subtractin $\quad \underline{x} - \underline{y} = \underline{\underline{F}}(\underline{X} - \underline{Y})$ or

$$\boxed{\varphi(\underline{X}) = \varphi(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y})}$$

Like a Taylor series but for hom. def. this is true even if $|\underline{X} - \underline{Y}|$ is not small.

Given $\quad \underline{g}(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$ and

$\underline{d}_i(\underline{X}) = \underline{X} + \underline{a}_i \qquad i = 1, 2$

$$(\underline{d}_1 \circ \underline{g})(\underline{X}) = \underline{d}_1(\underline{g}(\underline{X})) = \underline{g}(\underline{X}) + \underline{a}_1$$

$$= \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{a}_1$$

choose $\underline{a}_1 = \underline{\varphi}(\underline{Y}) - \underline{Y}$, translation of fixedpoint.

note $\underline{\varphi}$ itself does not have a fixedpoint!

substitute

$$(\underline{d}_1 \circ \underline{g})(\underline{X}) = \cancel{\underline{Y}} + \underline{\underline{F}}(\underline{X} - \underline{Y}) + \underline{\varphi}(\underline{Y}) - \cancel{\underline{Y}}$$

$$= \underline{\varphi}(\underline{Y}) + \underline{\underline{F}}(\underline{X} - \underline{Y}) = \underline{\varphi}(\underline{X})$$

$$\Rightarrow \quad \underline{\varphi}(\underline{X}) = (\underline{d}_1 \circ \underline{g})(\underline{X}) \; \checkmark$$

$\Rightarrow$ always extract translation and assume that our def. has a fixed point.

Strech-rotation decomposition

Let $\underline{\varphi}(\underline{X})$ be hom. def. with fixed point $\underline{Y}$ so that $\underline{\varphi}(\underline{X}) = \underline{Y} + \underline{\underline{F}}(\underline{X}-\underline{Y})$ then we have

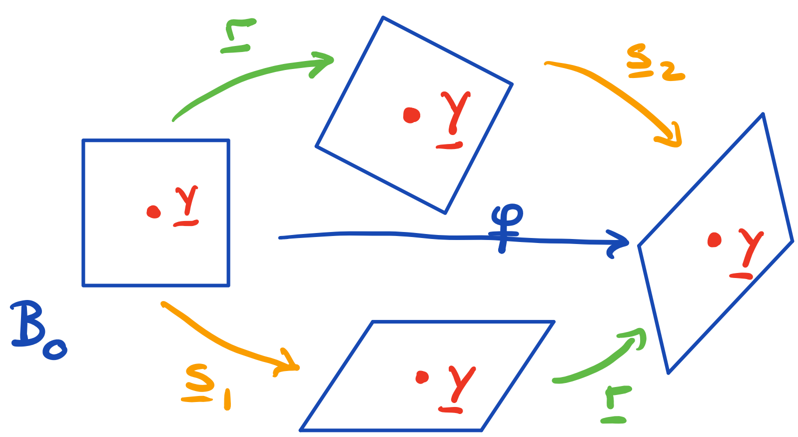
$$\boxed{\underline{\varphi} = \underline{r} \circ \underline{s}_1 = \underline{s}_2 \circ \underline{r}}$$

where $\underline{r} = \underline{Y} + \underline{\underline{R}}(\underline{X}-\underline{Y})$ is a rotation around $\underline{Y}$

$$\left.\begin{aligned} \underline{s}_1 &= \underline{Y} + \underline{\underline{U}}(\underline{X}-\underline{Y}) \\ \underline{s}_2 &= \underline{Y} + \underline{\underline{V}}(\underline{X}-\underline{Y}) \end{aligned}\right\} \text{ streches from } \underline{Y}$$

The tensors $\underline{\underline{R}}$, $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T\underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are given by polar decomposition

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$$



To see this consider

$$(\underline{r} \circ \underline{s}_1)(\underline{X}) = \underline{r}(\underline{s}_1(\underline{X})) = \underline{Y} + \underline{\underline{R}}(\underline{s}_1(\underline{X}) - \underline{Y})$$

$$= \underline{Y} + \underline{\underline{R}}(\cancel{\underline{Y}} + \underline{\underline{U}}(\underline{X} - \underline{Y}) - \cancel{\underline{Y}})$$

$$= \underline{Y} + \underline{\underline{R}}\,\underline{\underline{U}}(\underline{X} - \underline{Y}) =$$

$$= \underline{Y} + \underline{\underline{F}}(\underline{X} - \underline{Y})$$

$$(\underline{r} \circ \underline{s}_1)(\underline{X}) = \varphi(\underline{X}) \quad ✓$$

for $s_2 \circ r = \varphi$ see PS!

Strech tensors

Both $\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ and $\underline{\underline{V}} = \sqrt{\underline{\underline{F}}\,\underline{\underline{F}}^T}$ are s.p.d.

$\Rightarrow$ spectral decomposition

$$\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i \quad \text{and} \quad \underline{\underline{V}} = \sum_{i=1}^{3} \lambda_i \, \underline{v}_i \otimes \underline{v}_i$$

where $\{\lambda_i, \underline{u}_i\}$ and $\{\lambda_i, \underline{v}_i\}$ are eigenpairs of $\underline{\underline{U}}$ & $\underline{\underline{V}}$

same eigenvalues but different eigenvectors.

Note: $\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}} \rightarrow \underline{\underline{R}}^T \underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}} \rightarrow \underline{\underline{U}} = \underline{\underline{R}}^T \underline{\underline{V}}\,\underline{\underline{R}}$

Consider char. polynomial

$$p_u(\lambda) = \det(\underline{\underline{U}} - \lambda \underline{\underline{I}}) = \det(\underline{\underline{R}}^T \underline{\underline{V}}\, \underline{\underline{R}} - \lambda \underline{\underline{R}}^T \underline{\underline{R}})$$

$$= \det(\underline{\underline{R}}^T (\underline{\underline{V}} - \lambda \underline{\underline{I}}) \underline{\underline{R}}) = \cancel{\det(\underline{\underline{R}}^T)}^{1} \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) \cancel{\det(\underline{\underline{R}})}^{1}$$

$$= \det(\underline{\underline{V}} - \lambda \underline{\underline{I}}) = p_v(\lambda)$$

$\Rightarrow$ $\underline{\underline{U}}$ and $\underline{\underline{V}}$ have same eigenvalues

λ_i's are principal streches

$\underline{u}_i$ and $\underline{v}_i$ are right and left principal dir.

The λ_i's give the streching of the body in the $\underline{u}_i$ and $\underline{v}_i$ directions.

What is the relation between $\underline{u}_i$ and $\underline{v}_i$?

$$\underline{\underline{U}}\, \underline{u}_i = \lambda_i \underline{u}_i$$

$$\underline{\underline{R}}\, \underline{\underline{U}}\, \underline{u}_i = \lambda_i \underline{\underline{R}}\, \underline{u}_i \qquad F = \underline{\underline{R}}\, \underline{\underline{U}} = \underline{\underline{V}}\, \underline{\underline{R}}$$

$$\underline{\underline{V}}\, \underbrace{\underline{\underline{R}}\, \underline{u}_i}_{\underline{v}_i} = \lambda_i \underbrace{\underline{\underline{R}}\, \underline{u}_i}_{\underline{v}_i}$$

$\boxed{\underline{v}_i = \underline{\underline{R}}\, \underline{u}_i}$ differ by rotation.

In summary:

Any hom. def. φ can be decomposed into a sequence of 3 elementary deformations:

1) Translation
2) Rotation
3) Strech along principal directions

Example: $\varphi = \underline{s}_2 \circ \underline{r} \circ \underline{d}_2$

$\varphi = \underline{r} \circ \underline{s}_1 \circ \underline{d}_2$

...

Note: These results for hom. def. hold for any def. in a small neighborhood by Taylor expansion.