Local Lagrangian form of balance laws

Consider a body with reference configuration B undergoing a motion $\underline{\varphi}(\underline{X},t)$. Denote the current configuration $B_t = \underline{\varphi}_t(B)$. Consider an arbitrary subset Ω_t of B_t and let Ω be the corresponding subset of B, so that $\Omega_t = \underline{\varphi}(\Omega)$.

The Lagrangian balance laws, in terms of $\underline{X}$, can be obtained from the Eulerian balance laws, in terms of $\underline{x}$, simply by change of variable. Here we we develop the Lagrangian balance laws directly from the integral form of the balance laws.

I) Balance of mass

We already derived this

$$\boxed{\rho_m(\underline{X},t)\det \underline{\underline{F}}(\underline{X},t) = \rho_0(\underline{X})} \qquad \text{for all } \underline{X} \in B,\ t \geq 0$$

The mass density is a known quantity in the Lagrangian formulation.

II) Balance of linear momentum

The integral balance law is

$$\frac{d}{dt}\underline{L}[\Omega_t] = \underline{r}[\Omega_t]$$

where
$$\underline{L}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\,\underline{v}(\underline{x},t)\,dV_x$$
$$\underline{r}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\,\underline{b}(\underline{x},t)\,dV_x + \int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{n}\,dA_x$$

change variable of integrals from $\underline{x}$ to $\underline{X}$

$$\underline{L}[\Omega_t] = \int_{\Omega} \rho_m(\underline{X},t)\,\underbrace{\underline{v}_m(\underline{X},t)}_{V(\underline{X},t)=\dot{\underline{\varphi}}(\underline{X},t)}\,\det\underline{\underline{F}}(\underline{X},t)\,dV_X$$

$$= \int_{\Omega} \rho_0(\underline{X},t)\,\dot{\underline{\varphi}}(\underline{X},t)\,dV_X$$

To change variables on the r.h.s. we need Nanson's formula $\quad \underline{n}\,dA_x = \det\underline{\underline{F}}\,\underline{\underline{F}}^{-T}\underline{N}\,dA_X$

we have

$$\underline{r}[\Omega_t] = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\,\underline{n}\, dA_x + \int \rho\,\underline{b}\, dV_x$$

$$= \int_{\partial\Omega} \underline{\underline{\sigma}}_m \det \underline{\underline{F}}\; \underline{\underline{F}}^{-T} \underline{N}\, dA_X + \int \rho_m b_m \det \underline{\underline{F}}\, dV_X$$

To simplify the notation we introduce the tensor

$$\boxed{\underline{\underline{P}}(\underline{X},t) = \det \underline{\underline{F}}(\underline{X},t)\; \underline{\underline{\sigma}}_m(\underline{X},t)\; \underline{\underline{F}}(\underline{X},t)^{-T}}$$ first Piola-Kirchhoff stress tensor

so that

$$\underline{r}[\Omega_t] = \int_{\partial\Omega} \underline{\underline{P}}\, \underline{N}\, dA_X + \int_{\Omega} \rho_o\, b_m\, dV_X$$

substituting into the integral balance we have

$$\frac{d}{dt}\int_{\Omega} \rho_o \dot{\underline{\varphi}}\, dV_X = \int_{\partial\Omega} \underline{\underline{P}}\,\underline{N}\, dA_X + \int_{\Omega} \rho_o b_m\, dV_X$$

since $\rho_o \neq \rho_o(t)$ and Ω is constant we have

$$\int_{\Omega} \rho_o \ddot{\underline{\varphi}}\, dV_X = \int_{\Omega} \nabla_X \cdot \underline{\underline{P}} + \rho_o b_m\, dV_X$$

where we have used the div. thm.. By the arbitrariness of Ω we have

$$\boxed{\rho_o \ddot{\underline{\varphi}} = \nabla_X \cdot \underline{\underline{P}} + \rho_o \underline{b}_m}$$ local Lagrangian form

Note: $\underline{\underline{P}}$ is the natural stress tensor in material description in that it relates the traction of a surface to its normal: $$\boxed{\begin{aligned} \underline{t}(\underline{x},t) &= \underline{\underline{\sigma}}(\underline{x},t)\,\underline{n} \\ \underline{T}(\underline{X},t) &= \underline{P}(\underline{X},t)\,\underline{N} \end{aligned}}$$

Here $\underline{T}$ is the (nominal) Piola-kirchhoff traction vector and $\underline{t}$ is the (true) Cauchy traction vector.

The resultant force on any surface element is

$$d\underline{f} = \underline{t}\, dA_x = \underline{T}\, dA_X$$

hence $\underline{T}$ points in same direction as $\underline{t}$.

III) Balance of Angular momentum

From the Cauchy stress we have $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

and the definiton $\underline{P} = J\, \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$ we have

$$\underline{\underline{\sigma}} = \frac{1}{J}\, \underline{\underline{P}}\, \underline{\underline{F}}^T = \underline{\underline{\sigma}}^T = \frac{1}{J}\, \underline{\underline{F}}\, \underline{\underline{P}}^T$$

$$\Rightarrow \quad \boxed{\underline{\underline{P}}\, \underline{\underline{F}}^T = \underline{\underline{F}}\, \underline{\underline{P}}^T}$$

$\underline{\underline{P}} \neq \underline{\underline{P}}^T$ hence $\underline{P}$ has 9 independent components.

Motivates definition of a second Piola-Kirchhoff stress

$$\boxed{\underline{\underline{\Sigma}} = \underline{\underline{F}}\, \underline{\underline{P}}} \qquad \text{so that} \qquad \underline{\underline{\Sigma}} = \underline{\underline{\Sigma}}^T$$

Characterization of Networking

Networking is defined as the external power that is not converted into kinetic energy

$$W[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt} K[\Omega_t]$$

Derive a relation between rate of change in kinetic energy and the power of external and internal forces

From the integral balance laws:

$$\mathcal{K}[\Omega_t] = \int_{\Omega_t} \frac{1}{2} \rho |\underline{v}|^2 \, dV_x$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega_t} \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega} \underline{\underline{\sigma}} \, \underline{v} \cdot \underline{n} \, dA_x$$

where we have used $\underline{v} \cdot \underline{\underline{\sigma}} \, \underline{n} = \underline{\underline{\sigma}}^T \underline{v} \cdot n = \underline{\underline{\sigma}} \, \underline{v} \cdot \underline{n}$

Changing variables we have

$$\mathcal{K}[\Omega_t] = \int_{\Omega} \frac{1}{2} \rho_0 |\dot{\underline{\varphi}}|^2 \, dV_X$$

$$\mathcal{P}[\Omega_t] = \int_{\Omega} \rho_0 \underline{b}_m \cdot \underline{v}_m \, dV_X + \int_{\partial\Omega} J \, \underline{\underline{\sigma}}_m \underline{v}_m \cdot \underline{\underline{F}}^{-T} \underline{N} \, dA_X$$

$$= \int_{\Omega} \rho_0 \underline{b}_m \cdot \underline{v}_m \, dV_X + \int_{\partial\Omega} J \, \underbrace{\underline{v}_m}_{\dot{\underline{\varphi}}} \cdot \underbrace{\underline{\underline{\sigma}}_m \underline{\underline{F}}^{-T}}_{\underline{\underline{P}}} \underline{N} \, dA_X$$

$$= \int_\Omega \rho_0 \, \underline{b}_m \cdot \dot{\underline{\varphi}} \, dV_X + \int_{\partial\Omega} \dot{\underline{\varphi}} \cdot \underline{\underline{P}} \, \underline{N} \, dA_X$$

$$= \int_\Omega \rho_0 \, \underline{b}_m \cdot \dot{\underline{\varphi}} \, dV_X + \int_{\partial\Omega} \underline{\underline{P}}^T \dot{\underline{\varphi}} \cdot \underline{N} \, dA_X \qquad \underline{\underline{P}} \neq \underline{\underline{P}}^T \, !$$

applying the div. thm.

$$= \int_\Omega \rho_0 \, \underline{b}_m \cdot \dot{\underline{\varphi}} + \nabla_X \cdot (\underline{\underline{P}}^T \dot{\underline{\varphi}}) \, dV_X$$

using the identity $\nabla_X \cdot (\underline{\underline{P}}^T \dot{\underline{\varphi}}) = (\nabla_X \cdot P) \cdot \dot{\underline{\varphi}} + \underline{\underline{P}} : \nabla \dot{\underline{\varphi}}$

and identifying $\nabla \dot{\underline{\varphi}} = \dot{\underline{\underline{F}}}$ we have the following expression for the power of external forces

$$\boxed{\mathcal{P}[\Omega_t] = \int_\Omega [\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m] \cdot \dot{\underline{\varphi}} + \underline{\underline{P}} : \dot{\underline{\underline{F}}} \quad dV_X}$$

The rate of change of kinetic energy becomes

$$\frac{d}{dt} K[\Omega_t] = \int_\Omega \frac{1}{2} \rho_0 \frac{d}{dt} |\underline{v}|^2 \, dV_X = \int_\Omega \dot{\underline{\varphi}} \cdot (\rho_0 \ddot{\underline{\varphi}}) \, dV_X$$

using linear momentum balance $\rho \ddot{\varphi} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m$

$$\frac{d}{dt} K[\Omega_t] = \int_\Omega (\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m) \cdot \dot{\underline{\varphi}} \, dV_X$$

$$= \mathcal{P}[\Omega_t] - \int_\Omega \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_X$$

so that the Lagrangian definition of net working is

$$\boxed{W[\Omega_t] = \int_{\Omega} \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_X}$$

this is analogous to the Eulerian definition of stress power. However, $\underline{\underline{\sigma}} : \underline{\underline{d}}$ measures power per unit volume of B_t while $\underline{\underline{P}} : \dot{\underline{\underline{F}}}$ measures power per unit volume of B.

Local Lagrangian form of 1st Law of Thermo.

Integral balance law

$$\frac{d}{dt}\mathcal{U}[\Omega_t] = Q[\Omega_t] + \mathcal{W}[\Omega_t]$$

where internal energy is

$$\mathcal{U}[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, u(\underline{x},t)\, dV_x$$

$$= \int_{\Omega} \rho_m(\underline{X},t)\, u_m(\underline{X},t)\, J(\underline{X},t)\, dV_X$$

$$= \int \rho_o(X)\, U(\underline{X},t)\, dV_X$$

where $\rho_o(X) = \rho_m(X,t)\, J(\underline{X},t)$ and $U(\underline{X},t) = u_m(X,t)$

Rate of net heating

$$Q[\Omega_t] = \int_{\Omega_t} \rho(\underline{x},t)\, r(\underline{x},t)\, dV_x - \int_{\partial\Omega_t} \underline{q}(\underline{x},t)\cdot \underline{n}(\underline{x})\, dA_x$$

use Nansen's formula: $\underline{n}\, dA_x = J \underline{\underline{F}}^{-T} \underline{N}\, dA_X$

$$Q[\Omega_t] = \int_{\Omega_0} \rho_0 R(\underline{X},t)\, dV_X - \int_{\partial\Omega_0} q_m(\underline{X},t)\cdot(J\underline{\underline{F}}^{-T}\underline{N})\, dA_X$$

where $R(\underline{X},t) = r_m(\underline{X},t)$

property of transpose: $q_m \cdot \underline{\underline{F}}^{-T}\underline{N} = \underline{\underline{F}}^{-1} q_m \cdot N$

$$= \int_{\Omega} \rho_0 R\, dV_X - \int_{\partial\Omega} J \underline{\underline{F}}^{-1} q_m \cdot \underline{N}\, dA_X$$

introduce: $\underline{Q} = J\underline{\underline{F}}^{-1} q_m$ material heat flux

$$\Rightarrow \quad Q[\Omega_t] = \int_{\Omega_0} \rho_0 R\, dV_X - \int_{\partial\Omega_0} \underline{Q}\cdot\underline{N}\, dA_X$$

subsituting into 1st law + div. thm + localization

$$\boxed{\rho_0 \dot{\underline{U}} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot Q + \rho_0 R}$$

local Lagrangian form of 1st law

Local Lagrangian form of the second law

From integral balance

$$\int_{\Omega_t} \rho \dot{s}\, dV_x \geq \int_{\Omega_t} \frac{\rho r}{\theta}\, dV_x - \int_{\partial\Omega} \frac{q \cdot n}{\theta}\, dA_x$$

change to Ω

$$\int_{\Omega_0} \rho_0 \dot{S}\, dV_X \geq \int_{\Omega_0} \frac{\rho_0 R}{\Theta}\, dV_X - \int_{\partial\Omega_0} \frac{\underline{Q} \cdot \underline{N}}{\Theta}\, dA_X$$

where $\Theta = \theta_m \qquad S = s_m$

using div. thm. and localization

$$\boxed{\rho_0 \dot{s}_m \geq \frac{\rho_0 R}{\Theta} - \nabla_X \cdot \left(\frac{\underline{Q}}{\Theta} \right)}$$

Clausius-Duhem in Lagrangian form

Introduce Helmholtz free energy

$$\Psi(\underline{X}, t) = U(\underline{X}, t) - \Theta(\underline{X}, t)\, S(\underline{X}, t)$$

similar to Eulerian case

$$\rho_0 \dot{\Psi} \leq \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \rho_0 S \dot{\Theta} - \frac{1}{\Theta} \underline{Q} \cdot \nabla_X \Theta$$

Reduced C-D inequality

Summary of Lagrangian formulation

Lagrangian balance laws

kinematic:	$\underline{V} = \dot{\underline{\varphi}}$	3
lin. mom.:	$\rho_0 \dot{\underline{V}} = \nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{b}_m$	3
ang. mom.:	$\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T$	3
energy:	$\rho_0 \dot{U} = \underline{\underline{P}} : \dot{\underline{\underline{F}}} - \nabla_X \cdot \underline{Q} + \rho_0 R$	+ 1
		10

Lagrangian fields:

$\underline{\varphi}$	$\underline{V}$	$\underline{\underline{P}}$	Θ	$\underline{Q}$	U	
3	3	9	1	3	1	= 20

We have 20 unknown and 10 equation

$\Rightarrow$ one less than the Eulerian formutation

because the density is known!

Again we need 10 additional constraints!

Remarks:

1) In many situations $\underline{V}$ is not needed explicitly. $\Rightarrow$ 17 unknowns & 7 eqns

2) Need constitutive equations that relate $\underline{\underline{\Sigma}} = \underline{\underline{F}}^{-1}\underline{\underline{P}}$, $\underline{Q}$, $\underline{U}$ to φ and Θ

3) When thermal effects are also neglected $\Rightarrow$ 15 unknowns & 6 equations

System closure requires const. eqns relating $\underline{\underline{\Sigma}}$ to φ

$$\underline{ds}^T \cdot \underline{ds} = (\underline{n}\, ds)^T \cdot (\underline{n}\, ds) = \underline{n}^T \cdot \underline{n}\, ds^2 = ds^2$$

$$= (J \underline{F}^{-T} \underline{N}\, dS)^T \cdot (J \underline{\underline{F}}^{-T} \underline{N}\, dS)$$

$$= J^2 dS^2 \quad \underline{N}^T \cdot F^{-1} F^{-T} \underline{N}$$