Local Eulerian Balance Laws

Consider a body with reference configuration B undergoing a motion $\underline{\varphi}(\underline{X},t)$. Denote the current configuration $B_t = \underline{\varphi}_t(B)$. Consider an arbitrary subset Ω_t of B_t and let Ω be the corresponding subset of B, so that $\Omega_t = \underline{\varphi}(\Omega)$.

I) Conservation of mass

From the integral form $\frac{d}{dt} M[\Omega_t] = 0$ we have that $M[\Omega_t] = M[\Omega]$, using the transformation of volume integrals we have

$$M[\Omega] = \int_{\Omega_t} \rho(\underline{x},t)\, dV_x = \int_{\Omega} \rho_m(\underline{X},t) \det \underline{\underline{F}}(\underline{X},t)\, dV_X$$

where $\rho_m(\underline{X},t) = \rho(\underline{\varphi}(\underline{X},t),t)$.

At $t=0$, $\underline{x} = \underline{X}$, $\Omega_t = \Omega$ and $\det \underline{\underline{F}} = 1$, so that we have

$$M[\Omega] = \int_{\Omega_0} \rho(\underline{x},0)\, dV_x = \int_{\Omega} \rho(\underline{X},0)\, dV_X = \int_{\Omega} \rho_0(\underline{X})\, dV_X$$

where $\rho_0(\underline{X}) = \rho(\underline{X},0)$.

Conservation of mass requires

$$\int_\Omega \left[\rho_m(\underline{X},t)\det\underline{\underline{F}}(\underline{X},t) - \rho_0(\underline{X})\right] dV_X = 0$$

by arbitrariness of Ω we have

$$\boxed{\rho_m(\underline{X},t)\det\underline{\underline{F}}(\underline{X},t) = \rho_0(\underline{X})}$$

Lagrangian statement of mass conservation.

To convert this to Eulerian form we take $\frac{\partial}{\partial t}$

$$\underbrace{\frac{\partial}{\partial t}\rho_m(\underline{X},t)}_{\dot{\rho}(\underline{x},t)}\det\underline{\underline{F}}(\underline{X},t) + \rho_m(\underline{X},t)\underbrace{\frac{\partial}{\partial t}\det\underline{\underline{F}}(\underline{X},t)}_{\det\underline{\underline{F}}\,(\nabla_x\cdot\underline{v})_m} = 0$$

dividing by $\det\underline{\underline{F}}$ and switching to spatial description

$$\dot{\rho}(\underline{x},t) + \rho(\underline{x},t)\,\nabla_x\cdot\underline{v} = 0$$

$$\Rightarrow \quad \boxed{\dot{\rho} + \rho\,\nabla_x\cdot\underline{v} = 0} \quad \text{local Eulerian form}$$

expanding the material derivative we have

$$\frac{\partial}{\partial t}\rho + \underbrace{\nabla_x\rho\cdot\underline{v} + \rho\,\nabla_x\cdot\underline{v}}_{\nabla_x\cdot(\rho\underline{v})} = 0$$

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla_x\cdot(\rho\underline{v}) = 0} \quad \underline{\text{conservative}}\ \text{local Eulerian form}$$

Time derivative of integrals relative to mass

$$\boxed{\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\,dV_x = \int_{\Omega_t} \dot{\phi}(\underline{x},t)\rho(\underline{x},t)\,dV_x}$$

where $\phi(\underline{x},t)$ is any spatial scalar, vector or tensor field.

$$\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\,dV_x = \int_{\Omega} \phi_m(\underline{X},t)\underbrace{\rho_m(\underline{X},t)\det\underline{\underline{F}}(\underline{X},t)}_{\rho_0(\underline{X})}\,dV_X$$

$$\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\,dV_x = \int_{\Omega} \phi_m(\underline{X},t)\rho_0(\underline{X})\,dV_X$$

Take derivative

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \phi(\underline{x},t)\rho(\underline{x},t)\,dV_x &= \int_{\Omega} \frac{d}{dt}\phi_m(\underline{X},t)\,\rho_0(X)\,dV_X \\ &= \int_{\Omega} \dot{\phi}_m(\underline{X},t)\,\rho_0(\underline{X})\,dV_X \\ &= \int_{\Omega} \dot{\phi}_m(\underline{X},t)\,\rho_m(\underline{X},t)\det\underline{\underline{F}}(\underline{X},t)\,dV_X \\ &= \int_{\Omega_t} \dot{\phi}(\underline{x},t)\,\rho(\underline{x},t)\,dV_x \quad \checkmark\end{aligned}$$

II) Balance of linear momentum

For an arbitrary $\Omega_t \subseteq B_t$ we have

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{t}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

where $\rho, \underline{v}, \underline{t}$ and $\underline{b}$ are spatial fields.

Cauchy stress field: $\quad \underline{t} = \underline{\underline{\sigma}}\, \underline{n}$

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{\underline{\sigma}}\, \underline{n}\, dA_x + \int_{\Omega_t} \rho \underline{b}\, dV_x$$

using tensor divergence theorem

$$\frac{d}{dt}\int_{\Omega_t} \rho \underline{v}\, dV_x = \int_{\Omega_t} \nabla\cdot\underline{\underline{\sigma}} + \rho \underline{b}\, dV_x$$

using derivative relative to mass

$$\int_{\Omega_t} \rho \dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} - \rho\underline{b}\, dV_x = 0$$

by the arbitrariness of Ω_t, we have

$$\boxed{\rho \dot{\underline{v}} - \nabla\cdot\underline{\underline{\sigma}} = \rho \underline{b}} \quad \text{local Eulerian form}$$

Also referred to as Cauchy's first equation of motion.

To rewrite this in conservative form consider the following

$$\rho \dot{\underline{v}} = \rho \frac{\partial \underline{v}}{\partial t} + \rho (\nabla_x \underline{v}) \underline{v} = \frac{\partial}{\partial t}(\rho \underline{v}) - \frac{\partial \rho}{\partial t} \underline{v} + (\nabla_x \underline{v})(\rho \underline{v})$$

using mass balance $\frac{\partial \rho}{\partial t} = - \nabla_x \cdot (\rho \underline{v})$

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v}) \underline{v} + (\nabla_x \underline{v})(\rho \underline{v})$$

using $\nabla \cdot (\underline{a} \otimes \underline{b}) = (\nabla \underline{a}) \underline{b} + \underline{a} \, \nabla \cdot \underline{b}$ (see HW5 Q4)

$$\rho \dot{\underline{v}} = \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v})$$

Hence we have conservative local Eulerian form

$$\boxed{\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla_x \cdot (\rho \underline{v} \otimes \underline{v} - \underline{\underline{\sigma}}) = \rho \underline{b}}$$

conserved quantity: $\rho \underline{v}$ = linear momentum

advective mom. flux: $\rho \underline{v} \otimes \underline{v}$

diffusive mom. flux: $- \underline{\underline{\sigma}}$

III) Balance of angular momentum

For an arbitrary $\Omega_t \subseteq B_t$ we have

$$\frac{d}{dt}\int_{\Omega_t} \underline{x} \times \rho\, \underline{v}\, dV_x = \int_{\partial\Omega_t} \underline{x} \times \underline{t}\, dA_x + \int_{\Omega_t} \underline{x} \times \rho \underline{b}\, dV_x$$

The left hand side becomes

$$\frac{d}{dt}\int_{\Omega_t} \rho(\underline{x} \times \underline{v})\, dV_x = \int_{\Omega_t} \rho \frac{d}{dt}(\underline{x} \times \underline{v})\, dV_x =$$

$$= \int_{\Omega_t} \rho(\dot{\underline{x}} \times \underline{v} + \underline{x} \times \dot{\underline{v}})\, dV_x \qquad \dot{\underline{x}} = \underline{v}$$

$$= \int_{\Omega_t} \rho(\underline{x} \times \dot{\underline{v}})\, dV_x \qquad \underline{v} \times \underline{v} = \underline{0}$$

Substituting Cauchy stress field the r.h.s becomes

$$\int_{\Omega_t} \rho(\underline{x} \times \dot{\underline{v}})\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\, \underline{n}\, dA_x + \int_{\Omega_t} \rho(\underline{x} \times \underline{b})\, dV_x$$

$$\int_{\Omega_t} \underline{x} \times (\rho\dot{\underline{v}} - \rho b)\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\, \underline{n}\, dA_x$$

substitute linear mom. balance $\rho\dot{\underline{v}} - \rho\underline{b} = \nabla_x \cdot \underline{\underline{\sigma}}$

$$\boxed{\int_{\Omega_t} \underline{x} \times \nabla_x \cdot \underline{\underline{\sigma}}\, dV_x = \int_{\partial\Omega} \underline{x} \times \underline{\underline{\sigma}}\, \underline{n}\, dA_x}$$

This is exactly the statement we had for the static case

in Lecture 10 on Mechanical Equilibrium.

$\Rightarrow$ $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ extends to transient cases.