Balance of energy and entropy in local Eulerian form

Before deriving local forms of First and Second laws, we derive a relation between the rate of change of Kinetic energy and the power of external and internal forces.

Net working in Eulerian form

Power: $\mathcal{P} = \underline{f} \cdot \underline{v}$

Newton's 2^{nd} law: $\underline{f} = m\underline{a} \rightarrow \underline{f} = \frac{d}{dt}(m\underline{v}) = m\dot{\underline{v}}$

Start by taking dot product of $\underline{v}$ and lin. mom. balance

$$\rho \dot{\underline{v}} \cdot \underline{v} = \rho \underline{v} \cdot \dot{\underline{v}} = (\nabla_x \cdot \underline{\underline{\sigma}}) \cdot \underline{v} + \rho \underline{b} \cdot \underline{v}$$

integrating over an arbitrary $\Omega_t \subseteq B_t$ (to identify K, P, W)

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} (\nabla_x \cdot \underline{\underline{\sigma}}) \cdot \underline{v} + \rho \underline{b} \cdot \underline{v} \, dV_x$$

use identity $\nabla \cdot (\underline{\underline{A}}^T \underline{b}) = (\nabla \cdot \underline{\underline{A}}) \cdot \underline{b} + \underline{\underline{A}} : \nabla \underline{b}$ (Lecture 4)

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \nabla_x \underline{v} + \nabla \cdot (\underline{\underline{\sigma}}^T \underline{v}) + \rho \underline{b} \cdot \underline{v} \, dV_x$$

Using property $\underline{\underline{S}} : \underline{\underline{D}} = \underline{\underline{S}} : \mathrm{sym}(\underline{\underline{D}})$ if $\underline{\underline{S}} = \underline{\underline{S}}^T$ we can introduce the rate of strain tensor $\underline{\underline{d}} = \mathrm{sym}(\nabla_x \underline{v}) = \frac{1}{2}(\nabla_x \underline{v} + \nabla_x \underline{v}^T)$.

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \underline{\underline{d}} + \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega_t} \underline{\underline{\sigma}} \underline{v} \cdot \underline{n} \, dA_x$$

where we have used tensor divergence thm.

Using the definition of transpose $\underline{\underline{\sigma}} \underline{v} \cdot \underline{n} = \underline{v} \cdot \underline{\underline{\sigma}}^T \underline{n} = \underline{v} \cdot \underline{t}$

$$\int_{\Omega_t} \rho \underline{v} \cdot \dot{\underline{v}} \, dV_x = \int_{\Omega_t} -\underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x + \underbrace{\int_{\Omega_t} \rho \underline{b} \cdot \underline{v} \, dV_x + \int_{\partial\Omega_t} \underline{t} \cdot \underline{v} \, dA_x}_{\mathcal{P}[\Omega_t]}$$

Now we can identify the left hand side as

$$\frac{d}{dt} \mathcal{K}[\Omega_t] = \frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \rho \, \underline{v} \cdot \underline{v} \, dV_x = \frac{1}{2} \int_{\Omega_t} \rho \frac{d}{dt} (\underline{v} \cdot \underline{v}) \, dV_X$$

$$\frac{d}{dt}(v_i v_i) = \dot{v}_i v_i + v_i \dot{v}_i = 2(v_i \dot{v}_i)$$

$$\frac{d}{dt} \mathcal{K}[\Omega_t] = \int_{\Omega_t} \rho \, \underline{v} \cdot \dot{\underline{v}} \, dV_x$$

so that we have the result

$$\boxed{\frac{d}{dt} \mathcal{K}[\Omega_t] + \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x = \mathcal{P}[\Omega_t]}$$

by comparison with $\mathcal{W}[\Omega_t] = \mathcal{P}[\Omega_t] - \frac{d}{dt} \mathcal{K}[\Omega_t]$

$$\Rightarrow \boxed{\mathcal{W}[\Omega_t] = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x}$$

The quantity $\underline{\underline{\sigma}} : \underline{\underline{d}}$ is called the <u>stress power</u> associated with a motion. It corresponds to the rate of work done by internal forces (stresses) in a continuum body.

Local Eulerian form of First Law

The integral form of First Law

$$\frac{d}{dt}\mathcal{U}[\Omega_t] = Q[\Omega_t] + \mathcal{W}[\Omega_t]$$

where $\mathcal{U}[\Omega_t] = \int_{\Omega_t} \rho u \, dV_x$

$$Q[\Omega_t] = \int_{\Omega_t} \rho r \, dV_x - \int_{\partial\Omega_t} \underline{q} \cdot \underline{n} \, dA_x$$

$$\mathcal{W}[\Omega_t] = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x$$

Hence we have

$$\frac{d}{dt} \int_{\Omega_t} \rho u \, dV_x = \int_{\Omega_t} \underline{\underline{\sigma}} : \underline{\underline{d}} \, dV_x - \int_{\partial\Omega_t} \underline{q} \cdot \underline{n} \, dA_x + \int_{\Omega_t} \rho r \, dV_x$$

using derivative relative to mass and divergence Thm

$$\int_{\Omega} (\rho \dot{u} - \underline{\underline{\sigma}} : \underline{\underline{d}} + \nabla_x \cdot \underline{q} + \rho r) \, dV_x$$

by the arbitraryness of Ω_t we have

$$\boxed{\rho \dot{u} = \underline{\underline{\sigma}} : \underline{\underline{d}} - \nabla_x \cdot \underline{q} + \rho r} \quad \text{local Eulerian form}$$

To write it in conservative form we expand

expand the material time derivative and use

the balance of mass

$$\rho \dot{u} = \rho\left(\frac{\partial u}{\partial t} + \nabla_x u \cdot \underline{v}\right) = \frac{\partial}{\partial t}(\rho u) - u\frac{\partial \rho}{\partial t} + \rho \nabla_x u \cdot \underline{v}$$

$$= \frac{\partial}{\partial t}(\rho u) + u\, \nabla_x \cdot (\rho \underline{v}) + \nabla_x u \cdot (\rho \underline{v})$$

$$= \frac{\partial}{\partial t}(\rho u) + \nabla_x \cdot (\underline{v} \rho u)$$

Substituting into the local form and collecting the flux terms we have

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \nabla_x \cdot \left[\underline{v}\rho u + \underline{q}\right] = \underline{\underline{\sigma}} : \underline{\underline{d}} + \rho r}$$

conservative local Eulerian form

Local Eulerian Form of the Second Law

The integral form of the Clausius-Duhem form of the the Second Law is

$$\frac{d}{dt}\int_{\Omega_t} \rho s \, dV_x \geq \int_{\Omega_t} \frac{\rho r}{\theta} \, dV_x - \int_{\partial\Omega_t} \frac{q \cdot n}{\theta} \, dA_x$$

After applying the Divergence Thm and invoking the arbitrariness of Ω_t we have

$$\boxed{\rho \dot{s} \geq \rho r/\theta - \nabla_x \cdot (q/\theta)}$$

Clausius-Duhem inequality in local Eulerian form

After multiplying by θ and expanding the divergence

$$\theta \rho \dot{s} \geq \rho r - \nabla_x \cdot q + \theta^{-1} q \cdot \nabla_x \theta$$

Which can be written as

$$\boxed{\delta - \theta^{-1} q \cdot \nabla_x \theta \geq 0}$$

where $\delta = \theta \rho \dot{s} - (\rho r - \nabla_x \cdot q)$ is the internal dissipation density per unit volume. Difference between local entropy increase and the local heating.

Note:

I) Any point where $\nabla_x \theta = 0$ the dissipation is non-negative, $\delta \geq 0$. $\Rightarrow$ bodies with homogeneous θ have non-neg. dissipation.

II) If $\delta = 0$, i.e. a reversible process, then $q \cdot \nabla_x \theta \leq 0$.

Thus q is at an angle > 90 from $\nabla_x \theta$.

$\Rightarrow$ heat flows down the temperature gradient.

To study the consequences of Clausius-Duhem inequality for constitutive laws we introduce the field

$$\boxed{\psi(\underline{x},t) = \phi(\underline{x},t) - \theta(\underline{x},t)\, s(\underline{x},t)}$$

Helmholtz free energy density. This is the portion of the free energy available for performing work at const. θ.

$\Rightarrow$ Reformulate Clausius-Duhem in terms of ψ

Material derivative of free energy

$$\frac{d}{dt}(\theta s) = \frac{\partial}{\partial t}(\theta s) + \nabla_x(\theta s)\cdot\underline{v} = \theta\frac{\partial s}{\partial t} + s\frac{\partial \theta}{\partial t} + \theta\nabla_x s\cdot\underline{v} + s\nabla_x\theta\cdot\underline{v}$$

$$= \theta\left(\frac{\partial s}{\partial t} + \nabla_x s\cdot\underline{v}\right) + s\left(\frac{\partial \theta}{\partial t} + \nabla_x\theta\cdot\underline{v}\right)$$

$$= \theta\dot{s} + s\dot{\theta}$$

from definition of $\dot{\psi}$

$$\dot{\psi} = \dot{u} - \theta\dot{s} - \dot{\theta}s \quad \Rightarrow \quad \dot{u} = \dot{\psi} + \theta\dot{s} + s\dot{\theta}$$

substituting into local form of 1st law

$$\rho\dot{u} = \underline{\underline{\sigma}}:\underline{\underline{d}} - \nabla_x\cdot q + \rho r$$

$$\rho\dot{\psi} + \rho\theta\dot{s} + \rho s\dot{\theta} = \underline{\underline{\sigma}}:\underline{\underline{d}} - \nabla_x\cdot q + \rho r$$

$$\rho\theta\dot{s} = \underline{\underline{\sigma}}:\underline{\underline{d}} - \nabla_x\cdot q + \rho r - \rho\dot{\psi} - \rho s\dot{\theta}$$

substituting into 2nd law

$$\theta\rho\dot{s} \geq \rho r - \nabla_x\cdot q + \theta^{-1} q\cdot\nabla_x\theta$$

$$\underline{\underline{\sigma}}:\underline{\underline{L}} - \cancel{\nabla_x\cdot q} + \cancel{\rho r} - \rho\dot{\psi} - \rho s\dot{\theta} \geq \cancel{\rho r} - \cancel{\nabla_x\cdot q} + \theta^{-1} q\cdot\nabla_x\theta$$

solve for $\rho\dot{\psi}$

$$\boxed{\rho\dot{\psi} \leq \underline{\underline{\sigma}}:\underline{\underline{d}} - \rho s\dot{\theta} - \theta^{-1} q\cdot\nabla_x\theta}$$

This is called the reduced Clausius-Duhem inequality, because it is independent of local heat supply r and heat flux, q, if $\nabla_x \Theta = 0$. $\Rightarrow$ homogeneous bodies of classical thermo.

Note:

In a homogeneous body, $\nabla\Theta = 0$, we have that

$$\rho\dot{\phi} \leq \underline{\underline{\sigma}} : \underline{\underline{d}}$$

for a reversible process this becomes an equality.

$\Rightarrow$ rate of change of free energy is equal to the stress power.