Cauchy-Green Strain Tensor

Consider a deformation $\varphi : \mathcal{B} \to \mathcal{B}'$ with $\underline{\underline{F}} = \nabla \varphi$, then the (right) Cauchy-Green strain tensor is

$$\boxed{\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}} \, .$$

Note that $\underline{\underline{C}}$ is always symmetric pos. definite.

The deformation gradient $\underline{\underline{F}}$ contains information about both rotations and streches. Using the right polar decomposition we have

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$$

$\underline{\underline{R}}$ is rotation matrix

$\underline{\underline{U}} = \sqrt{\underline{\underline{F}}^T \underline{\underline{F}}}$ is right strech tensor

Clearly $\underline{\underline{C}} = \underline{\underline{U}}^2$ and the rotation $\underline{\underline{R}}$ implicit in $\underline{\underline{F}}$ is not present in in $\underline{\underline{C}}$.

$\Rightarrow$ The right Cauchy Green strain tensor only contains information about streches.

Hence we can cannot obtain $\underline{\underline{F}}$ from $\underline{\underline{C}}$!

Remarks:

1) Stricktly the right-stretch tensor $\underline{\underline{U}}$ is sufficient. We introduce $\underline{\underline{C}} = \underline{\underline{U}}^2$ to avoid the tensor square root.

Simple example:

$$[\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$[\underline{\underline{C}}] = [\underline{\underline{F}}^T][\underline{\underline{F}}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

To get $[\underline{\underline{U}}]$ we need to solve eigenvalue problem

$$\begin{vmatrix} 1-\mu & 0 & 0 \\ 0 & 5-\mu & 4 \\ 0 & 4 & 5-\mu \end{vmatrix} = (1-\mu)(5-\mu)^2 - 16(1-\mu) = 0$$

Eigenvalues: $\mu_{1,2} = 1 \quad \mu_3 = 9$

Eigenvectors: $[\underline{u}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [\underline{u}_2] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \underline{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Hence: $[\underline{\underline{U}}] = \sqrt{[C]} = \sum_{i=1}^{3} \sqrt{\mu_i}\, \underline{u}_i \otimes \underline{u}_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

2) $\underline{\underline{U}} = \sum_{i=1}^{3} \lambda_i \, \underline{u}_i \otimes \underline{u}_i$ where

λ_i's are principal stretches

$\underline{u}_i$'s are right principal directions

$\underline{\underline{C}} = \underline{\underline{U}}^2 = \sum_{i=1}^{3} \lambda_i^2 \, \underline{u}_i \otimes \underline{u}_i$

$\mu_i = \lambda_i^2$ eig. values of $\underline{\underline{C}}$ are squares of principal strecthes

eigenvectors are right principal dir.

3) $C_{KL} = F_{iK} F_{iL}$ "material strain tensor"

spatial indices are contracted

Other strain tensors

I) $\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$: Green-Lagrange tensor

$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})$ material tensor $\Rightarrow$ linear theory

II) $\underline{\underline{b}} = \underline{\underline{F}}\,\underline{\underline{F}}^T = \underline{\underline{V}}^2$: Left Cauchy-Green tensor

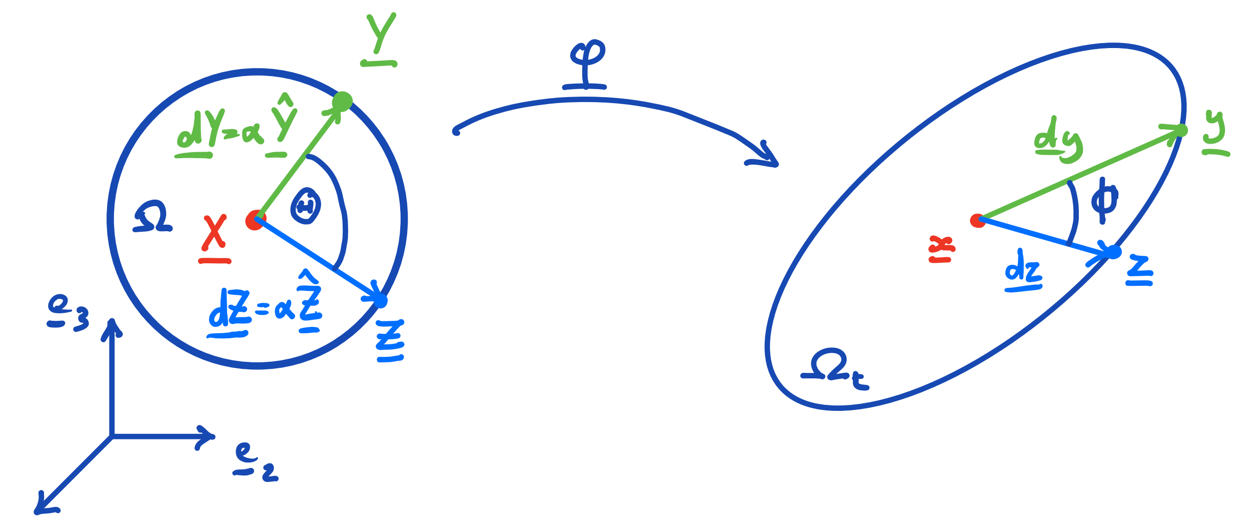
$b_{kl} = F_{kI} F_{lI}$ "spatial tensor" (a.k.a. Finger tensor)

III) $\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1})$: Euler-Almansi tensor

$e_{kl} = \frac{1}{2}(\delta_{kl} - F^{-1}_{Ik} F^{-1}_{Il})$ "spatial tensor"

Interpretation of $\underline{\underline{C}}$

How are changes in relative position and orientation of material points quantified by $\underline{\underline{C}}$?



Consider spherical domain Ω with radius $\alpha > 0$ around $\underline{X}$. Given two unit vectors $\underline{\hat{Y}}$ and $\underline{\hat{Z}}$ consider the points $\underline{Y} = \underline{X} + \alpha\underline{\hat{Y}} = \underline{X} + d\underline{Y}$ and $\underline{Z} = \underline{X} + \alpha\underline{\hat{Z}} = \underline{X} + d\underline{Z}$.

Let $\underline{x}, \underline{y}$ and $\underline{z}$ denote the corresponding points Ω' with $\phi \in [0, \pi]$ the angle between the vectors $d\underline{y} = \underline{y} - \underline{x}$ and $d\underline{z} = \underline{z} - \underline{x}$.

Cauchy-Green strain relations

For any point $\underline{X} \in B$ and unit vectors $\hat{\underline{Y}}$ and $\hat{\underline{Z}}$ we define $\lambda(\hat{\underline{Y}}) > 0$ and $\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) \in [0, \pi]$ by

$$\boxed{\lambda(\hat{\underline{Y}}) = \sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}} \quad \text{and} \quad \boxed{\cos\theta(\hat{\underline{Y}}, \hat{\underline{Z}}) = \frac{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}{\sqrt{\hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}}\,\sqrt{\hat{\underline{Z}} \cdot \underline{\underline{C}}\,\hat{\underline{Z}}}}}$$

I. Streches

In the limit as $\alpha \to 0$ we have

$$\frac{|\underline{y} - \underline{x}|}{|\underline{Y} - \underline{X}|} = \frac{|d\underline{y}|}{|d\underline{Y}|} \to \lambda(\hat{\underline{Y}}) \quad \text{and} \quad \frac{|\underline{z} - \underline{x}|}{|\underline{Z} - \underline{X}|} = \frac{|d\underline{z}|}{|d\underline{Z}|} \to \lambda(\hat{\underline{Z}})$$

Therefore $\lambda(\hat{\underline{Y}})$ is the strech in direction $\hat{\underline{Y}}$ at $\underline{X}$.

A stretch is the ratio of deformed to initial length.

To determine the stretch we use $d\underline{y} = \underline{\underline{F}}(\underline{X})\,d\underline{Y}$.

$$|d\underline{y}|^2 = d\underline{y} \cdot d\underline{y} = \underline{\underline{F}}\,d\underline{Y} \cdot (\underline{\underline{F}}\,d\underline{Y}) = d\underline{Y} \cdot \underline{\underline{F}}^T \underline{\underline{F}}\,d\underline{Y} = d\underline{Y} \cdot \underline{\underline{C}}\,d\underline{Y}$$
$$= \alpha^2\, \hat{\underline{Y}} \cdot \underline{\underline{C}}\,\hat{\underline{Y}}$$

$|d\underline{Y}|^2 = \alpha^2$ by definition

So that $\frac{|d\underline{y}|^2}{|d\underline{Y}|^2} = \hat{\underline{Y}} \cdot \underset{=}{C}\, \hat{\underline{Y}} = \lambda^2(\hat{\underline{Y}})$

taking square root: $\lambda(\underline{e}) = \sqrt{\hat{\underline{Y}} \cdot \underset{=}{C}\, \hat{\underline{Y}}}$ ✓

If $\underline{u}_i$ is a right-principal strech, so that

note: $\hat{\underline{U}}_i$ is the eigenvector of $\underset{=}{C}$ it is capitalized because it is a material vector

$$(\underset{=}{C} - \lambda_i^2 \underset{=}{I})\, \hat{\underline{U}}_i = 0 \qquad \text{(no sum)}$$

$$\hat{\underline{U}}_i \cdot \underset{=}{C}\, \hat{\underline{U}}_i - \lambda_i^2\, \hat{\underline{U}}_i \cdot \hat{\underline{U}}_i = 0 \qquad \hat{\underline{U}}_i \cdot \underset{=}{C}\, \hat{\underline{U}}_i = \lambda_i^2$$

then $\lambda(\hat{\underline{U}}_i) = \lambda_i$ which justifies referring to λ_i's as principal streches.

Arguments similar to determination of principal stresses show that $\lambda(\hat{\underline{Y}})$ has extremum if $\hat{\underline{Y}} = \hat{\underline{U}}_1$.

II. Shear

The shear $\gamma(\hat{\mathbf{Y}}, \hat{\mathbf{Z}})$ at $\mathbf{X}$ is the change in angle between the two directions $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$

$$\boxed{\gamma(\hat{\mathbf{Y}}, \hat{\mathbf{Z}}) = \Theta(\hat{\mathbf{Y}}, \hat{\mathbf{Z}}) - \theta(\hat{\mathbf{Y}}, \hat{\mathbf{Z}})}$$

where $\Theta(\mathbf{e}, \mathbf{d})$ is the angle between $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ in the reference configuration and $\theta(\hat{\mathbf{Y}}, \hat{\mathbf{Z}})$ is the angle between the deformed line segments $\mathbf{y}$ and $\mathbf{z}$ in the limit $\alpha \to 0$ so that

$$\boxed{\cos\phi \longrightarrow \cos\theta(\hat{\mathbf{Y}}, \hat{\mathbf{Z}})}$$

To see this consider $\qquad \cos\phi = \dfrac{d\mathbf{y} \cdot d\mathbf{z}}{|d\mathbf{y}|\,|d\mathbf{z}|}$

where
$$\begin{aligned} d\mathbf{y} \cdot d\mathbf{z} &= (\mathbf{F}\, d\mathbf{Y}) \cdot (\mathbf{F}\, d\mathbf{Z}) \\ &= d\mathbf{Y} \cdot \mathbf{F}^T \mathbf{F}\, d\mathbf{Z} = d\mathbf{Y} \cdot \mathbf{C}\, d\mathbf{Z} \\ &= \alpha^2\, \hat{\mathbf{Y}} \cdot \mathbf{C} \hat{\mathbf{Z}} \end{aligned}$$

with $|d\mathbf{y}| = \alpha\sqrt{\hat{\mathbf{Y}} \cdot \mathbf{C} \hat{\mathbf{Y}}}$ and $|d\mathbf{z}| = \alpha\sqrt{\hat{\mathbf{Z}} \cdot \mathbf{C} \hat{\mathbf{Z}}}$

so that $\cos\phi = \dfrac{d\hat{\underline{Y}} \cdot \underline{\underline{C}}\, d\hat{\underline{Z}}}{\sqrt{d\hat{\underline{Y}} \cdot \underline{\underline{C}}\, d\hat{\underline{Y}}}\sqrt{d\hat{\underline{Z}} \cdot \underline{\underline{C}}\, d\hat{\underline{Z}}}} \xrightarrow[\alpha \to 0]{} \cos\theta(d\hat{\underline{Y}}, d\hat{\underline{Z}})$

Components of $\underline{\underline{C}}$

Let C_{IJ} be the components of $\underline{\underline{C}}$ in an arbitrary frame $\{\underline{e}_I\}$, then for any point $\underline{X} \in B$ we have that

$$\boxed{\begin{aligned} C_{II} &= \lambda^2(\underline{e}_I) \\ C_{IJ} &= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\sin\gamma(\underline{e}_I, \underline{e}_J) \end{aligned}} \quad \text{(no sum)}$$

$\Rightarrow$ The diagonal components of C are the squares of the streches in coord. directions. Off diagonal components are related to shears between coordinate directions.

The expression for the diagonal components follows directly from the first Cauchy-Green strain relation

$\lambda(\underline{Y}) = \sqrt{\underline{Y} \cdot \underline{\underline{C}}\,\underline{Y}}$ and $C_{II} = \underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_I$ (no sum)

so that $C_{II} = \lambda^2(\underline{e}_I)$. ✓

For the off-diagonal components C_{IJ} $(I \neq J)$ we start with the second Cauchy-Green strain relation

$$\cos\theta(\underline{e}_I, \underline{e}_J) = \frac{\underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_J}{\sqrt{\underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_I}\sqrt{\underline{e}_J \cdot \underline{\underline{C}}\,\underline{e}_J}} \quad \text{and} \quad C_{IJ} = \underline{e}_I \cdot \underline{\underline{C}}\,\underline{e}_J$$

so that

$$C_{IJ} = \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\cos\theta(\underline{e}_I, \underline{e}_J) .$$

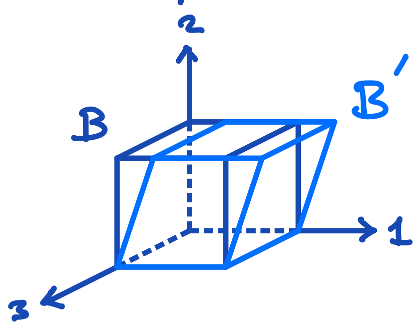
The shear between two basis vectors is

$$\gamma(\underline{e}_I, \underline{e}_J) = \underbrace{\Theta(\underline{e}_I, \underline{e}_J)}_{\frac{\pi}{2}} - \theta(\underline{e}_I, \underline{e}_J)$$

so that
$$\begin{aligned} C_{IJ} &= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\cos\left(\tfrac{\pi}{2} - \gamma(\underline{e}_I, \underline{e}_J)\right) \\ &= \lambda(\underline{e}_I)\,\lambda(\underline{e}_J)\sin\left(\gamma(\underline{e}_I, \underline{e}_J)\right) \end{aligned}$$ ✓

The components of $\underline{\underline{C}}$ directly quantify stretch and shear unlike the components of $\underline{\underline{E}}$.

Example: Simple shear



$$B = \{ \underline{X} \in \mathbb{E}^3 \mid 0 < X_i < 1 \}$$

$$\underline{x} = \underline{\varphi}(\underline{X}) = \begin{bmatrix} X_1 + \alpha X_2 \\ X_2 \\ X_3 \end{bmatrix} \qquad \alpha > 0$$

"simple shear in $\underline{e}_1$-$\underline{e}_2$ plane

Deformation gradient:

$$[\underline{\underline{F}}] = [\nabla \underline{\varphi}] = \begin{bmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ homogeneous deformation

Cauchy-Green strain tensor:

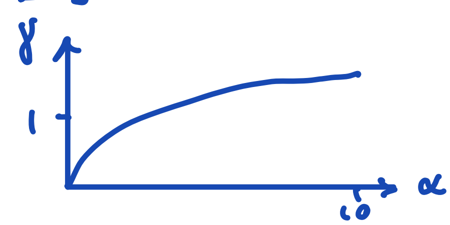
$$[\underline{\underline{C}}] = \underline{\underline{F}}^T \underline{\underline{F}} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & \alpha & 0 \\ \alpha & 1+\alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

Find the shear γ for direction pair $(\underline{e}_1, \underline{e}_2)$

$$\gamma(\underline{e}_1, \underline{e}_2) = \Theta(\underline{e}_1, \underline{e}_2) - \theta(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \theta(\underline{e}_1, \underline{e}_2)$$

$$\cos\theta(\underline{e}_1, \underline{e}_2) = \frac{[\underline{e}_1]^T [\underline{\underline{C}}] [\underline{e}_2]}{\sqrt{[\underline{e}_1]^T [\underline{\underline{C}}] [\underline{e}_1]} \sqrt{[\underline{e}_2]^T [\underline{\underline{C}}] [\underline{e}_2]}} = \frac{\alpha}{\sqrt{1}\sqrt{1+\alpha^2}}$$

$$\Rightarrow \underline{\underline{\gamma(\underline{e}_1, \underline{e}_2) = \frac{\pi}{2} - \operatorname{acos}\left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)}}$$



Find $\gamma(\underline{e}_1, \underline{e}_3)$ again $\Theta(\underline{e}_1, \underline{e}_3) = \frac{\pi}{2}$

$$\cos\Theta(\underline{e}_1, \underline{e}_3) = \frac{C_{13}}{\sqrt{C_{11}}\sqrt{C_{33}}} = \frac{0}{1 \cdot 1} = 0$$

$$\gamma(\underline{e}_1, \underline{e}_3) = \frac{\pi}{2} - \underbrace{\operatorname{acos} 0}_{\frac{\pi}{2}} = \underline{\underline{0}}$$

What are the extreme values of the strech and their directions? $\Rightarrow$ eigenvalues & vectors

$$\begin{vmatrix} 1-\lambda^2 & \alpha & 0 \\ \alpha & 1+\alpha^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{vmatrix} = 0$$

$$\lambda_1^2 = 1 + \frac{\alpha^2}{2} + \alpha\sqrt{1+\alpha^2/4} > 1$$
$$\lambda_2^2 = 1$$
$$\lambda_3^2 = 1 + \frac{\alpha^2}{2} - \alpha\sqrt{1+\alpha^2/4} < 1$$

Principal directions:

$[\underline{v}_1] = [\sqrt{1+\alpha^2/4} - \alpha/2, 1, 0]$

$[\underline{v}_2] = [0, 0, 1]$ (not normalized)

$[\underline{v}_3] = [\sqrt{1+\alpha^2/4} + \alpha/2, -1, 0]$

$\Rightarrow$ λ_1 is max strech in dir $\underline{v}_1$

λ_3 is min strech in dir $\underline{v}_3$

there is no strech in dir $\underline{e}_3$